MATH 2802
FINAL EXAM, VERSION B

Name		Section	

Please **read all instructions** carefully before beginning.

- There are 10 problems in the exam. Each problem is worth 10 and the maximum score on this exam is 100 points.
- **Read through** all the exam *before starting* to solve the problems. Start with subproblems that are easy/fast to solve.
- You have 170 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work. You may use the last two pages as scratch paper.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Georgia Tech Honor Code

Having read the Georgia Institute of Technology Academic Honor Code, I understand and accept my responsibility as a member of the Georgia Tech community to uphold the Honor Code at all times. In addition, I understand my options for reporting honor violations as detailed in the code.

__________________ Signature

__________________ Date

1. [2 points each] Circle **T** if the statement is always true and circle **F** if it is ever false. Let A be an $n \times n$ matrix with column vectors $v_1, v_2, \ldots, v_n$ and let $T(x) = Ax$.

a) **T** **F** If $\det(A) = -1$ then the columns of A are linearly dependent.

b) **T** **F** The volumen of the paralelepid formed by $v_1, \ldots, v_n$ equals $\det(A)$.

c) **T** **F** The range of T is the same subspace as $Col\, A$.

d) **T** **F** If $v_1, \ldots v_n$ are linearly dependent then T is not a one-to-one function.

e) **T** **F** If the dimension of $ColA$ is 3, then any 3 vectors in $Col\, A$ form a basis of $Col\, A$.

2. [2 points each] Circle **T** if the statement is always true and circle **F** if it is ever false. The matrix A is $n \times n$.

a) **T** **F** If the algebraic multiplicities of all eigenvalues of A sum up to n then A is diagonalizable.

b) **T** **F** If A is diagonalizable matrix, then A is invertible.

c) **T** **F** If v satisfies $Av = v$, then v is a steady-state vector of A.

d) **T** **F** If c is a scalar and v is a vector then $||cv|| = c||v||$.

e) **T** **F** If y is a vector in a subspace W then $proj_W(y) = 0$.

3. **a)** [4pts] Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $T(x) = Ax$ with $A = \begin{pmatrix} -1 & 2 & 1 \\ 2 & -4 & 0 \\ 3 & -6 & 0 \end{pmatrix}$.

The image of the unit vector e_1 is:

The image of the unit vector e_2 is:

The image of the unit vector e_3 is:

Is the transformation T onto? why or why not?

b) [2pts] Give the 2×2 matrix A_1 associated to a rotation in $\mathbf{R}^2$ by an angle of $135°$ degrees.

c) [2pts] Given the 2×2 matrix A_2 associated that projects vectors to the line $x_1 = x_2$.

d) [2pts] For this problem, use the matrices from **b)-c)**.
If $U : \mathbf{R}^2 \to \mathbf{R}^2$ is defined by $U(x) = A_1 A_2 x$, describe what the transformation U does to the unit vectors.

4. Let $A = \begin{pmatrix} -2 & 0 & -5 & -2 \\ -4 & 3 & -8 & -1 \\ 0 & 9 & 6 & 13 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 5/2 & 1 \\ 0 & 1 & 2/3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

a) [3pts] Find a basis for $NulA$

b) [2pts] Find a linear dependency among the columns of A.

c) [3pts] Write down 3 distinct ways to verify whether a matrix B is invertible.
(Hint: recall the invertible matrix theorem)

d) [2pts] What is the determinant of $B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$?

5. **a)** [4pts] Use the row reduction below to find the LU factorization of A.

$$A = \begin{pmatrix} 2 & 4 & -1 \\ -4 & -5 & 3 \\ 2 & -5 & -4 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 \\ 0 & 3 & 1 \\ 0 & -9 & -3 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

b) [4pts] Use row reduction to compute the inverse of $B = \begin{pmatrix} 8 & -4 & 0 \\ -4 & 8 & 0 \\ -2 & 0 & 2 \end{pmatrix}$.

c) [2pts] An economy sector has production matrix C. Use the Leontief formula

$$(1 - C)x = d$$

to find the production vector x required to meet a demand of $d = \begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix}$.

(Hint: Use that $(I - C)^{-1} = 10B^{-1}$)

6. Let $v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$.

a) [2pts] Is $\{v_1, v_2\}$ a basis of $W = Span\{v_1, v_2\}$? Justify.

b) [4pts] Is $\{v_1, v_2\}$ an orthogonal basis of W? If not, find an orthogonal basis $\{u_1, u_2\}$ of W.

c) [4pts] Find the distance between v_3 and the plane W.

7. Let $A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$:

a) [6pts] The eigenvalues of A are $4, 3$ and 2. Find an orthogonal decomposition of A.

b) [1pt] Classify the quadratic form $Q(x) = x^T A x$ (i.e. indefinite, negative semidefinite, etc.):

c) [1pt] The quadratic form $Q(x) = x^T A x$, subject to the constraint $x^T x = 1$, has maximum value:

d) [2pts] According to the SVD decomposition of $B = \begin{pmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{pmatrix}$,

$$B = \begin{pmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{pmatrix} \begin{pmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}.$$

the singular values of B are:

8. Celia has one hour to spend at the CRC, and she wants to jog, play handball, and ride a stationary bike. Jogging burns 13 calories per minute, handball burns 11, and cycling burns 7. She jogs twice as long as she rides the bike. She wants to participate in each of these activities in order to burn exactly 600 calories.

a) [4pts] Let x, y, z represent the minutes Celia takes in playing handball, jogging and cycling. Setup a system of 3 equations that represent the problem above.

b) [3pts] Write the system of equations above as a matrix equation $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = b$.

Is the matrix equation consistent?

c) [3pts] If the system were to be inconsistent, **describe** the procedure of least-squares applied to $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = b$ (do not perform any computations).

9. A mouse lives in a maze. At every hour it moves from the room where it stands to one of the adjacent rooms with equal probability.

a) [4 points] Find a steady-state vector for matrix $P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$.

b) [2 points] In the long run, is there a room where the mouse is more likely to be at a given time?

c) [2 points] Design a mouse maze that matches the transition matrix P.

d) [2 points] Let $Q = \begin{pmatrix} 1-a & b \\ a & 1-b \end{pmatrix}$ and find an eigenvector of Q with eigenvalue 1.

10. [2 points each] The following matrices are decomposed using different methods: LU-factorization, QR-factorization, diagonalization, orthogonal diagonalization, and decomposition according to singular values.

Write down which method is used in each of the cases.

a) $A = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 8 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$

b) $A = \begin{pmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{pmatrix} = \begin{pmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{pmatrix} \begin{pmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}$

c) $A = \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$

d) $A = \begin{pmatrix} 2 & 4 & -1 \\ -4 & -5 & 3 \\ 2 & -5 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

e) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{pmatrix}$

[Scratch work]

Scoring Table

Please do not write on this area.

1	2	3	4	5	6	7	8	9	10	Total

[Scratch work below this line]