Section 7.1

Diagonalization of symmetric matrices

Motivation: Diagonalization

How did we **recognize** diagonalizable matrices?

- They are already diagonal
- They have n distinct eigenvalues

  Quick to check: only if matrix is triangular
- The algebraic and geometric multiplicities are equal for all eigenvalues and they sum up to n.

New criterion: Verify if matrix is *symmetric*!

- Symmetric, e.g. $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$
- **Not** symmetric, e.g. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & -4 & 0 \\ 6 & 1 & -4 \\ 0 & 6 & 1 \end{pmatrix}$

Warm up: $u^T u$ vs uu^T

If u is a vector in $\mathbf{R}^n$ with entries $u^T = (u_1, u_2, \ldots, u_n)$, then

- $u^T u = u_1^2 + u_2^2 + \cdots u_n^2$ is *a scalar*.
- uu^T is **an $n \times n$ matrix**:

$$uu^T = \begin{pmatrix} u_1 u_1 & u_1 u_2 & \cdots & u_1 u_n \\ u_2 u_1 & u_2 u_2 & \cdots & u_2 u_n \\ \vdots & \vdots & \ddots & \vdots \\ u_n u_1 & u_n u_2 & \cdots & u_n u_n \end{pmatrix}$$

A projection matrix!

In fact, uu^T is the *standard matrix* for the transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ that projects onto the line spanned by u.

Warm up: Inverse of an orthonormal matrix

For **orthogonal matrices** Q, with *column vectors* $u_1, u_2, \dots, u_n$ we already know that

$$Q^T Q = \begin{pmatrix} u_1 \cdot u_1 & 0 & \cdots & 0 \\ 0 & u_2 \cdot u_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & u_n \cdot u_n \end{pmatrix}$$

so for **orthonormal matrices** Q

$$Q^T Q = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & 1 \end{pmatrix}$$

What is *the inverse of Q*?

Orthogonally diagonalizable

Definition

An $n \times n$ matrix A is **orthogonally diagonalizable** if $A = PDP^{-1}$ with D *diagonal* matrix and P an *orthonormal* matrix.

To stress the orthogonality of P we write $A = PDP^T$.

Avoiding errors

Computations using orthogonal matrices usually prevents numerical errors from accumulating.

Collection of eigenvalues = 'Spectral'

Spectral decomposition

If D has diagonal entries $\lambda_1, \ldots, \lambda_n$ and P has columns $u_1, \ldots, u_n$ then

$$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T$$

- Fancy way of expressing the change of variables and
- the fact that *principal axes* are only stretch/contracted
- Each of $u_i u_i^T$ **is a projection matrix!**

Why? We have to name each entry of the vectors $u_1, \ldots, u_n$.

1. Say $u_k^T = (u_{k1}, u_{k2}, \ldots u_{kn})$.
2. Start with *a simple case*: $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1$
   - Compare the (i,j)-th entry of $u_k u_k^T$: $u_{ki} u_{kj}$
   - with the (i,j)-th entry of PP^T: $\sum_{k=1}^n u_{ki} u_{kj}$
3. Challenge: If the *λ's are different*, how the entries of PDP^T change?

Poll

Paper-based Poll

In a piece of **paper with your name**, *hand to the instructor:*

If $P = \begin{pmatrix} u_{11} & u_{21} & u_{31} & u_{41} \\ u_{12} & u_{22} & u_{32} & u_{42} \\ u_{13} & u_{23} & u_{33} & u_{43} \\ u_{14} & u_{24} & u_{34} & u_{44} \end{pmatrix}$

Write down P^T and compute

- the $(2,3)$-th entry of PP^T
- the $(2,3)$-th entry of $\begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{14} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \end{pmatrix}$
- the $(2,3)$-th entry of $\begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \\ u_{24} \end{pmatrix} \begin{pmatrix} u_{21} & u_{22} & u_{23} & u_{24} \end{pmatrix}$

Example: Orthogonally diagonalizable

Example

Orthogonally diagonalize the matrix $A = \begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}$

its *charactheristic equation* is $-(\lambda - 7)^2(\lambda + 2) = 0$.

Find a basis for each λ-eigenspace:

For $\lambda = 7$: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} \right\}$ For $\lambda = -2$: $\left\{ \begin{pmatrix} -1 \\ -1/2 \\ 1 \end{pmatrix} \right\}$

A suitable P

Is the set of eigenvectors above already orthogonal? orthonormal?

$$A = P \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{pmatrix} P^{-1}$$

Example: Orthogonally diagonalizable
continued

Verify:

- $v_3 = \begin{pmatrix} -1 \\ -1/2 \\ 1 \end{pmatrix}$ is *already orthogonal* to $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix}$
- but $v_1 \cdot v_2 \neq 0$.

Tackle this: Use Gram-Schmidt

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix} - \frac{-1/2}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/4 \\ 1 \\ 1/4 \end{pmatrix}$$

And $u_3 = v_3$. **Then normalize!**

$$P = \begin{pmatrix} 1/\sqrt{2} & -1\sqrt{18} & -2/3 \\ 0 & 4\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1\sqrt{18} & 2/3 \end{pmatrix}, \qquad D = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Example: Spectral Decomposition

Example

Construct a *spectral decomposition* of the matrix A with orthogonal diagonalization

$$A = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 8 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

Solution: Then $A = 8u_1u_1^T + 3u_2u_2^T$, each matrix is

$$u_1u_1^T = \begin{pmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{pmatrix}$$

$$u_2u_2^T = \begin{pmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{pmatrix}$$

Check: $8u_1u_1^T + 3u_2u_2^T = \begin{pmatrix} 32/5 & 16/5 \\ 16/5 & 8/5 \end{pmatrix} + \begin{pmatrix} 3/5 & -6/5 \\ -6/5 & 12/5 \end{pmatrix} = A$

Definition

An $n \times n$ matrix is **symmetric** if $A = A^T$.

Theorem

An $n \times n$ matrix A is *orthogonally diagonalizable* if and only if A is *symmetric*.

The easy observation: Let $A = PDP^T$ with D diagonal and P orthonormal.

Just check A is symmetric, that is $A = A^T$:

$$(\underbrace{PDP^T}_{A}) = (P^T)^T D^T P^T = \underbrace{PDP^T}_{A}$$

The difficult part (omitted here) is: if $A = A^T$ then

an *orthogonal diagonalization do exists*.

Summary

Spectral Theorem for Symmetric matrices

An $n \times n$ *symmetric* matrix A has the **following properties**.

- A has n *real eigenvalues*, counting multiplicities
- For each eigenvalue,
  the dimension of the λ-eigenspaces equal the algebraic multiplicity.
- The eigenspaces are *mutually orthogonal!*

  eigenvectors corresponding to different eigenvalues are orthogonal.
- A is *orthogonally diagonalizable*.

Extra: Eigenspaces are mutually orthogonal

Symmetric matrices only

Eigenspaces are mutually orthogonal

What does it mean?

If v_1 and v_2 are eixgenvectors that correspond to distinct eigenvalues λ_1 and λ_2

then $v_1 \cdot v_2 = 0$.

Trick to see this: Find a way to show that $(\lambda_1 - \lambda_2)v_1 \cdot v_2 = 0$.

Why? We assumed that $\lambda_1 \neq \lambda_2$ so necessarily $v_1 \cdot v_2 = 0$.

Hint: Compute $v_1^T A v_2$ in two different 'orders'

- Symmetry is *important:* You'll have to sustitute $A = A^T$ at some point.