Section 7.2

Quadratic Forms

Motivation: Non-linear functions

The following functions are *not linear*

- $f(x_1, x_2) = x_1^2 + 2x_2x_3$
- $g(x_1, x_2) = x_1^2 + x_2^2$

but they have *'dot-product'* expressions:

$$g(x) = x^T x = x^T I x$$

And in general, $x^T A x$

- gets you **a scalar**,
- is a sum that includes *'cross-product'* terms ax_ix_j

Quadratic Forms

Definition

A **quadratic form** on $\mathbf{R}^n$ is a function $Q : \mathbf{R}^n \to \mathbf{R}$ that can be expressed as $Q(x) = x^T A x$ where A is an $n \times n$ *symmetric matrix*.

Example

If $A = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ then

$$Q(x) = 4x_1^2 + 3x_2^2$$

Example

If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ then

$$Q(x) = x_1^2 + 2x_2x_3$$

Quadratic Forms

Example

Let $Q(x) = 5x_1^2 + 3x_2^2 + 2x_3^2 - 4x_1x_2 + 8x_2x_3$

Find the *matrix of the quadratic form*.

A must be symmetric:

- The coefficients of x_i^2 go on the diagonal of A,
- (i, j)-th and (j, i)-th entries are equal and sum up to the coefficient of x_ix_j.

Then

$$A = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 3 & 4 \\ 0 & 4 & 2 \end{pmatrix}$$

Back to change of variables

A consequence of the spectral theorem for symmetric matrices

The principal axes theorem

Let A be $n \times n$ symmetric matrix.

Then there is an **orthogonal change of variable** $x = Py$ that transforms the quadratic form $x^T Ax$ into a quadratic form

$y^T Dy$ with no cross-product terms.

If $A = PDP^{-1}$ with $P^T = P^{-1}$ and D diagonal, then

$$x^T Ax = \underbrace{x^T P}_{y^T} \ D \ \underbrace{P^{-1} x}_{y}$$

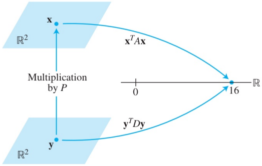


FIGURE 1 Change of variable in $\mathbf{x}^T A\mathbf{x}$.

Back to change of variables

continued

A consequence of the spectral theorem for symmetric matrices

The principal axes theorem

Let A be $n \times n$ symmetric matrix.

Then there is an *orthogonal change of variable* $x = Py$ that transforms the quadratic form $x^T Ax$ into a quadratic form
$y^T Dy$ with no cross-product terms.

- Columns of P are: **Principal axes**
- The *vector y* is the *coordinate vector of x*

  relative to the basis formed by the principal axes

Change of variables

Example

Make a change of variables that transforms the quadratic form

$$Q(x_1, x_2) = x_1^2 - 5x_2^2 - 8x_1x_2$$

into a quadratic form with *no cross-product* terms

General Formula: there is an *orthonormal matrix P* such that

$$A = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^T$$

the change of variables is given by $y = P^T x = P^{-1} x$.

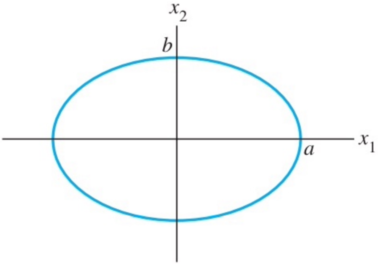
In this case, First $A = \begin{pmatrix} 1 & -4 \\ -4 & 5 \end{pmatrix}$, $\lambda_1 = 3, \lambda_2 = -7$ and $P = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

Then

$$y^T \begin{pmatrix} 3 & 0 \\ 0 & -7 \end{pmatrix} y = 3y_1^2 - 7y_2^2$$

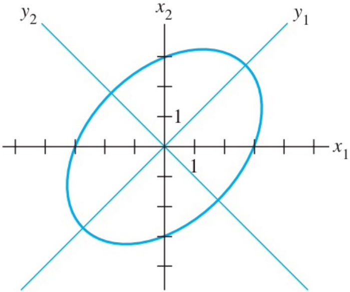
Geometric view: Contour curves

If $Q(x) = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2}$ then draw all points x for which $Q(x) = 1$.



$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \ a > b > 0$$

ellipse



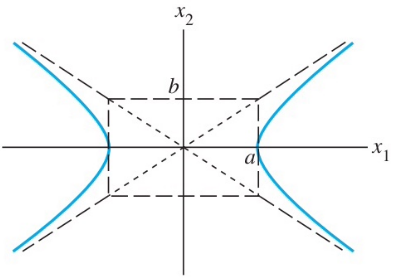
(a) $5x_1^2 - 4x_1x_2 + 5x_2^2 = 48$

To find *principal axes*, change variables

Standard position

Geometric view: Contour curves

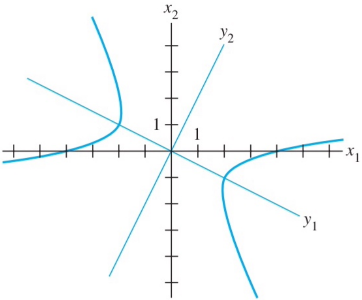
If $Q(x) = \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2}$ then draw all points x for which $Q(x) = 1$.



$$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1, \quad a > b > 0$$

hyperbola

Standard position



(b) $x_1^2 - 8x_1x_2 - 5x_2^2 = 16$

To find *principal axes*, change variables

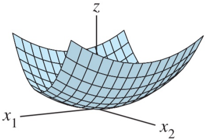
A quadratic form is

- *Indefinite:* if $Q(x)$ assumes **both** positive and negative values
- *Positive definite:* if $Q(x) > 0$ for all $x \neq 0$,
- *Negative definite:* if $Q(x) < 0$ for all $x \neq 0$,

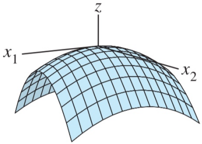
The prefix *semi* means e.g. $Q(x) \geq 0$ for all $x \neq 0$.

Eigenvalues

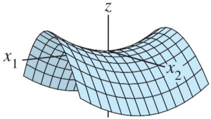
You can classify quadratic from knowing its eigenvalues (evaluate on principal axes)
e.g. Positive definite forms have *all eigenvalues* positive.



Positive definite



Negative definite

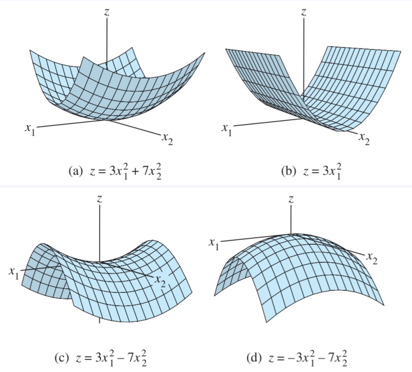


Indefinite

Poll

Paper-based poll

In a piece of **paper with your name**, *hand to the instructor*:
Find all indefinite quadratic forms among the display below



Only *d*) is indefinite, since *b*) does *not take negative* values, it is not indefinite.
The prefix **semi** means e.g. $Q(x) \geq 0$ for all $x \neq 0$.

Classification: do not jump to conclusions

False impression

All entries of A are positive, doesn't imply A is positive definite!

Example

Find a *vector x* such that $coloroliveQ(x) = x^T Ax < 0$, for $A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}$

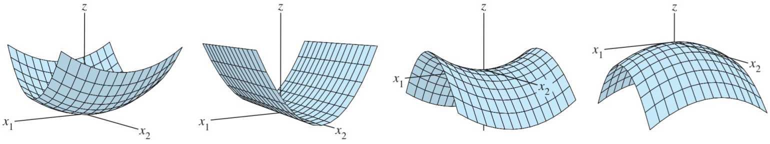
Solution: The eigenvalues of A are $5, 2, -1$.
Finding eigenvector for each eigenvalue $=$ finding the principal axes of $Q(x)$.
The orthonormal matrix is

$$P = \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}.$$

The vector for axis with eigenvalue -1 has $Q(x) = -1$; this is

$$v = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}.$$

Extra: All possible contour curves



(a) $z = 3x_1^2 + 7x_2^2$ (b) $z = 3x_1^2$ (c) $z = 3x_1^2 - 7x_2^2$ (d) $z = -3x_1^2 - 7x_2^2$

Positive Def.	**Negative Semidef.**	**Indefinite**	**Negative Def.**
Ellipses	Parallel lines	Hyperbolas	Empty
A point	A line	Two inters. lines	A point
Empty	Empty	Hyperbolas (Axes flipped)	Ellipses