Section 1.3

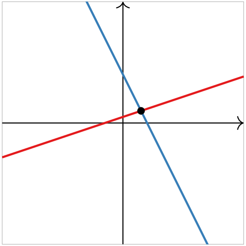
Vector Equations

Motivation

Linear algebra's *two viewpoints*:

- **Algebra**: systems of equations and their solution sets
- **Geometry**: intersections of points, lines, planes, etc.

$$\begin{aligned} x - 3y &= -3 \\ 2x + y &= 8 \end{aligned} \quad \rightsquigarrow$$

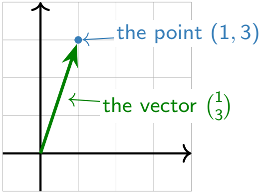


The **geometry** will give us *better insight into the properties* of systems of equations and their solution sets.

Vectors

Elements of $\mathbf{R}^n$ can be considered *points*...

or **vectors**:
arrows with a given
length and direction.



x-coordinate: *width* of vector *horizontally,*
y-coordinate: *height* of vector *vertically.*

It is *convenient* to express **vectors** in $\mathbf{R}^n$ **as matrices** with n rows and *one column*:

$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Note: Some authors use **bold typography** for vectors: $\mathbf{v}$.

Vector Algebra (applies to vectors in R^n)

Definition

- We can add two vectors together:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \\ c+z \end{pmatrix}.$$

- We can multiply, or **scale**, a vector by a real number:

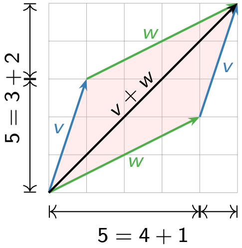
$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \cdot x \\ c \cdot y \\ c \cdot z \end{pmatrix}.$$

  Distinguish a vector from a real number: call c a **scalar**.
  $c\mathbf{v}$ is called a **scalar multiple** of $\mathbf{v}$.

For instance,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} \quad \text{and} \quad -2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}.$$

Addition: The parallelogram law



Geometrically, the sum of two vectors **v**,**w** is obtained by **creating a parallelogram**:

1. Place the tail of w at the head of v.
2. Sum vector $v + w$ has **tail**: tail of **v**
3. Sum vector $v + w$ has **head**: head of **w**

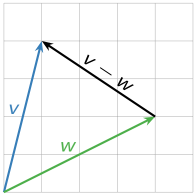
The width of $v + w$ is the sum of the widths, and likewise with the heights. For example,

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$$

Note: addition is commutative.

Geometry of vector substraction

If you add **v** − **w** to **w**, you get **v**.



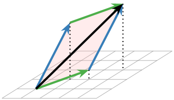
Geometrically, the difference of two vectors **v**,**w** is obtained as follows:

1. Place the tails of w and v at the *same point*.
2. Difference vector $v - w$ has **tail**: head of **w**
3. Difference vector $v - w$ has **head**: head of **v**

For example,

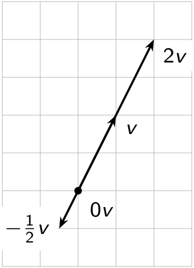
$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

This **works in higher dimensions** too!



Scalar multiples of a vector:
have the same *direction* but a different *length*.
The *scalar multiples* of **v** **form a line**.

Some multiples of v.



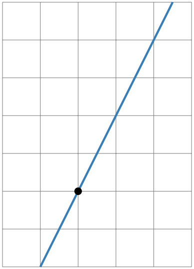
$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$2\mathbf{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$-\frac{1}{2}\mathbf{v} = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}$$

$$0\mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

All multiples of v.



Linear Combinations

We can *generate new vectors* with addition and scalar multiplication:

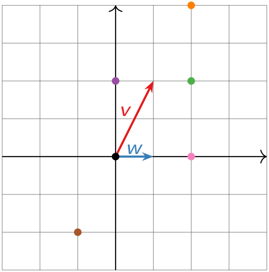
Definition

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

We call $\mathbf{w}$ a **linear combination** of the vectors $v_1, v_2, \ldots, v_p$, and the scalars $c_1, c_2, \ldots, c_p$ are called the **weights** or **coefficients**.

- $c_1, c_2, \ldots, c_p$ are scalars,
- $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are **vectors in** $\mathbf{R}^n$, and so is $\mathbf{w}$.

Example



Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

What are some linear combinations of v and w?

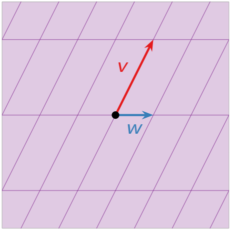
- $v + w$
- $v - w$
- $2v + 0w$
- $2w$
- $-v$

Poll

Poll

Is there any vector in $\mathbf{R}^2$ that is *not a linear combination* of v and w?

No: in fact, *every* vector in $\mathbf{R}^2$ is a combination of v and w.



(The purple lines are to help measure *how much of v and w* you need *to reach a given point*.)

Span

It will be important to handle *all linear combinations of a set* of vectors.

Definition

Let $v_1, v_2, \dots, v_p$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \dots, v_p$ is the collection of all linear combinations of $v_1, v_2, \dots, v_p$, and is denoted $\mathrm{Span}\{v_1, v_2, \dots, v_p\}$. In symbols:

$$\mathrm{Span}\{v_1, v_2, \dots, v_p\} = \left\{ x_1 v_1 + x_2 v_2 + \cdots + x_p v_p \;\middle|\; x_1, x_2, \dots, x_p \text{ in } \mathbf{R} \right\}.$$

In other words:

- $\mathrm{Span}\{v_1, v_2, \dots, v_p\}$ is the subset **spanned by** or **generated by** $v_1, v_2, \dots, v_p$.
- it's exactly the *collection of all b in* $\mathbf{R}^n$ such that the *vector equation* (unknowns $x_1, x_2, \dots, x_p$)

  $$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

  is consistent i.e., has a solution.

Poll

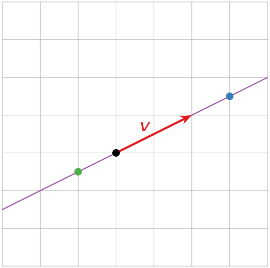
Poll

Which of the following are *possible shapes for the Span* $\{\mathbf{v}_1, \mathbf{v}_2\}$ of 2 vectors in $\mathbb{R}^3$?
Select all possible shapes!

- A Empty
- B Point
- C Line
- D Circle
- E the grid points on a 2-plane
- F the 4-plane

Answer: *B* and *C*.
(*Span is never empty*
and *two vectors may span a 2-plane*, but **not only** its grid points)

More Examples

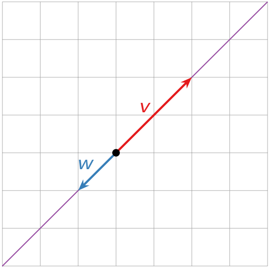


What are some linear combinations of $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

- $\frac{3}{2}v$
- $-\frac{1}{2}v$
- ...

What are *all* linear combinations of v?

All vectors cv for c a real number. I.e., all *scalar multiples* of v. These form a *line*.



Question

What are all linear combinations of

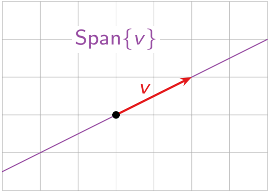
$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

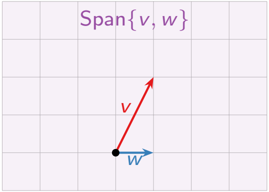
Answer: The line which contains both vectors.

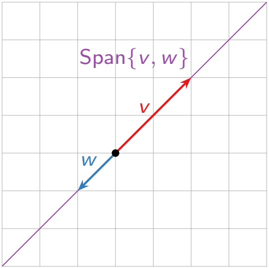
What's different about this example and the one on the poll?

Pictures of Span in R^2

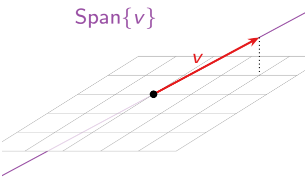
Drawing a picture of Span$\{v_1, v_2, \ldots, v_p\}$ is the same as drawing a picture of all linear combinations of $v_1, v_2, \ldots, v_p$.

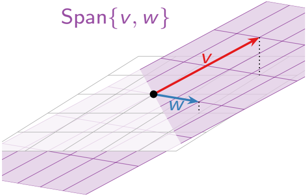


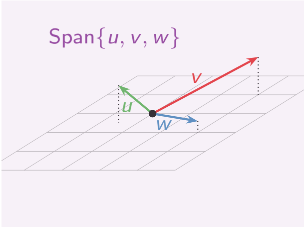


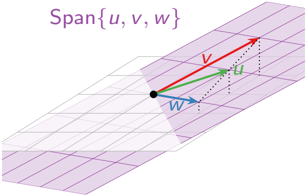


Pictures of Span in $\mathbf{R}^3$









Important

Even if *intuition and a geometric feeling* of what Span represents is important for class. You **will use the definition** of Span to solve problems on the exams.

Systems of Linear Equations

Question

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

This means: can we solve the equation

$$x \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

where x and y are the unknowns (the coefficients)? Rewrite:

$$\begin{pmatrix} x \\ 2x \\ 6x \end{pmatrix} + \begin{pmatrix} -y \\ -2y \\ -y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} x - y \\ 2x - 2y \\ 6x - y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

This is just a system of linear equations:

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3. \end{aligned}$$

Systems of Linear Equations

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3 \end{aligned} \quad \xrightarrow{\text{matrix form}} \quad \left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right)$$

$$\xrightarrow{\text{row reduce}} \quad \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\text{solution}} \quad \begin{aligned} x &= -1 \\ y &= -9 \end{aligned}$$

Conclusion:

$$-\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - 9\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

Systems of linear equations depend on the Span of a set of vectors!

Span of vectors and Linear equations

We have *three equivalent ways* to think about linear systems of equations:

Summary

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{b}$ be vectors in $\mathbf{R}^n$ and $x_1, x_2, \ldots, x_p$ be scalars.

1. A vector $\mathbf{b}$ is in the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.
2. The linear system with augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p & \mathbf{b} \\ | & | & & | & | \end{array}\right),$$

   is consistent ($\mathbf{v}_i$'s and $\mathbf{b}$ are the columns).
3. The vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$, has a solution.

Equivalent means that, for any given list of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{b}$, *either all three* statements are true, *or all three* statements are false.

Extra: So, what is *Span*?

To think about...

How many vectors are in $\text{Span}\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$?

A. Zero

B. One

C. Infinity

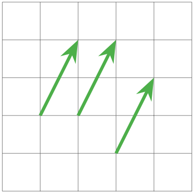
So far, it seems that $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is the smallest "linear space" (line, plane, etc.) containing **the origin** and all of the vectors $v_1, v_2, \ldots, v_p$.

We is made precise with 'vector subspace' definition.

Extra: Points and Vectors

So what is the difference between a point and a vector?

A vector need not start at the origin: *it can be located anywhere*! In other words, an arrow is determined by its length and its direction, not by its location.



These arrows all represent the vector $\begin{pmatrix}1\\2\end{pmatrix}$.

However, unless otherwise specified, we'll assume a vector starts at the origin: we'll usually be sloppy and identify the vector $\binom{1}{2}$ with the point $(1, 2)$.

This makes sense in the real world: many physical quantities, such as velocity, are represented as vectors. But it makes more sense to think of the velocity of a car as being located at the car.

Another way to think about it: a vector is a *difference* between two points, or the arrow from one point to another.

For instance, $\begin{pmatrix}1\\2\end{pmatrix}$ is the arrow from $(1, 1)$ to $(2, 3)$.