Section 1.4

The Matrix Equation $Ax = b$

Today: Spans and Solutions to Equations

Let $\mathbf{b} \in \mathbf{R}^n$ and A be a matrix with columns $v_1, v_2, \ldots, v_n \in \mathbf{R}^n$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

Very Important Fact That Will Appear on Every Midterm and the Final

$Ax = b$ has a solution

$$\iff \text{there exist } x_1, \ldots, x_n \text{ such that } A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b$$

"if and only if"

$$\iff \text{there exist } x_1, \ldots, x_n \text{ such that } x_1 v_1 + \cdots + x_n v_n = b$$

$$\iff b \text{ is a linear combination of } v_1, \ldots, v_n$$

$$\iff b \text{ is in the span of the columns of } A.$$

The last condition is geometric.

Matrix × Vector

the first number is the number of rows

the second number is the number of columns

Let A be an $m \times n$ matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad \text{with columns } v_1, v_2, \ldots, v_n$$

Definition

The **product** of A with a vector x in $\mathbf{R}^n$ is the linear combination

$$Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \overset{\text{def}}{=} x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

this means the equality is a *definition*

these must be equal

The output is a vector in $\mathbf{R}^m$.

Necessary: Number of columns of A equals number of rows of x.

Question

Let v_1, v_2, v_3 be vectors in $\mathbf{R}^3$. How can you write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

Answer: Let A be the matrix with colums v_1, v_2, v_3, and let x be the vector with entries $2, 3, -4$. Then

$$Ax = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 2v_1 + 3v_2 - 4v_3,$$

so the vector equation is equivalent to the matrix equation

$$Ax = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}.$$

Matrix × Vector

Another way

Definition

A *row vector* is a matrix with one row. The **product** of a row vector of length n and a (column) vector of length n **is a scalar!**

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \stackrel{\text{def}}{=} a_1x_1 + \cdots + a_nx_n.$$

If A is an $m \times n$ matrix with rows $r_1, r_2, \ldots, r_m$, and x is a vector in $\mathbf{R}^n$, then

$$Ax = \begin{pmatrix} — r_1 — \\ — r_2 — \\ \vdots \\ — r_m — \end{pmatrix} x = \begin{pmatrix} r_1x \\ r_2x \\ \vdots \\ r_mx \end{pmatrix}$$

This is **a vector in $\mathbf{R}^m$**.

Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} (4\ 5\ 6)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ (7\ 8\ 9)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 \\ 7 \cdot 1 + 8 \cdot 2 + 9 \cdot 3 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

Note this is the same as before:

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \\ 1 \cdot 7 + 2 \cdot 8 + 3 \cdot 9 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

Now you have **two ways of computing** Ax.

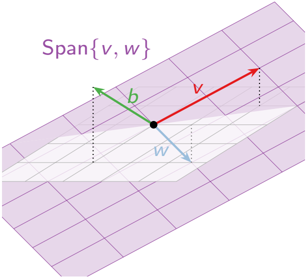
In the *second*, you calculate Ax *one entry at a time*.

Both are convenient, so we'll use both.

Spans and Solutions to Equations
Example 1

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?



Columns of A:

$$v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad w = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Output vector:

$$b = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

Is b contained in the span of the columns of A?

Conclusion: $Ax = b$ is *inconsistent*.

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?

Answer: Let's check by solving the matrix equation using row reduction.

The first step is to put the system into an augmented matrix.

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

The last equation is $0 = 1$, so the system is *inconsistent*.

In other words, the matrix equation

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

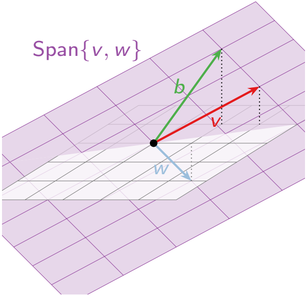
has no solution.

Spans and Solutions to Equations

Example 2

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?



Columns of A:

$$v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad w = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Solution vector:

$$b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Is b contained in the span of the columns of A? It looks like it: in fact,

$$b = 1v + (-1)w \implies x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?

Answer: Let's do this systematically using row reduction.

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

This gives us

$$x = 1 \qquad y = -1.$$

This is consistent with the picture on the previous slide:

$$1\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \qquad \text{or} \qquad A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Linear Systems, Vector Equations, Matrix Equations, . . .

Now have four equivalent ways of writing linear systems:

1. As a *system of equations*:

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5 \end{aligned}$$

2. As an *augmented matrix*:

$$\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right)$$

3. As a *vector equation* ($x_1 v_1 + \cdots + x_n v_n = b$):

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a *matrix equation* ($Ax = b$):

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

We will move back and forth freely between these over and over again.

When Solutions Always Exist

Here are **criteria** for a linear system **to always have a solution**.

Theorem

Let A be an $m \times n$ (non-augmented) matrix. The following are **equivalent**

1. $Ax = b$ has a *solution for all* b in $\mathbf{R}^m$.
2. The span of the columns of A is *all of* $\mathbf{R}^m$.
3. A has a pivot *in each row*.

recall that this means that for given A, either they're all true, or they're all false

Why is (1) the same as (3)?

Look at **reduced echelon** forms of A.

- If A has *a pivot in each row*:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{pmatrix} \quad \text{and } (\, A \mid b \,) \text{ reduces to:} \quad \left(\begin{array}{ccccc|c} 1 & 0 & \star & 0 & \star & \star \\ 0 & 1 & \star & 0 & \star & \star \\ 0 & 0 & 0 & 1 & \star & \star \end{array}\right).$$

  There's no b that makes it inconsistent, so there's *always a solution*.

- If A *doesn't have a pivot* in each row:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and this can be made inconsistent:} \quad \left(\begin{array}{ccccc|c} 1 & 0 & \star & 0 & \star & 0 \\ 0 & 1 & \star & 0 & \star & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{array}\right).$$