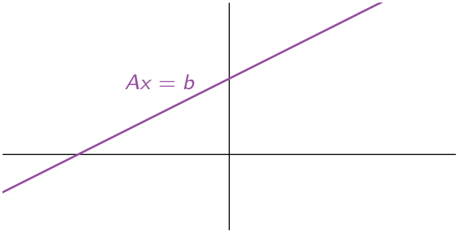
Section 1.5

Solution Sets of Linear Systems

Plan For Today

Describe and draw the solution set of $Ax = b$, using spans and parametric vector solutions.



Recall: the **solution set** is the collection of all vectors x such that $Ax = b$ is true.

Example 1

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}?$$

We know how to do this: first form an augmented matrix and row reduce.

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

The only solution is the trivial solution $x = 0$.

Observation

Since the last column (*everything to the right of the* $=$) was zero to begin, it will always *stay zero!*

For these cases it's not necessary to write augmented matrices.

Example 2, explained

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

$$\left(\begin{array}{cc|c} 1 & -3 & -3 \\ 2 & -6 & -6 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & -3 & -3 \\ 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\text{equation}} \quad x_1 - 3x_2 = -3$$

$$\xrightarrow{\text{parametric form}} \quad \begin{cases} x_1 = 3x_2 - 3 \\ x_2 = x_2 + 0 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

Note that p is itself a solution: take $x_2 = 0$.

Example 3, explained

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equation}} x_1 - 3x_2 = 0$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 3x_2 \\ x_2 = x_2 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Parametric vector forms

These equations are called the **parametric vector form** of the solutions.

It is obtained by listing equations for all the variables, in order, including the free ones, and making a vector equation.

Parametric Vector Form and Span

In general

Let A be an $m \times n$ matrix. If the *free variables* in the equation $Ax = b$ are $x_i, x_j, x_k, \ldots$

And the **parametric vector form** of the solution is

$$x = b' + x_i v_i + x_j v_j + x_k v_k + \cdots$$

for some vectors $b', v_i, v_j, v_k, \ldots$ in $\mathbf{R}^n$, and any scalars $x_i, x_j, x_k, \ldots$

Then the solution set is

$$b' \quad + \quad \mathrm{Span}\{v_i,\ v_j,\ v_k, \ldots\}.$$

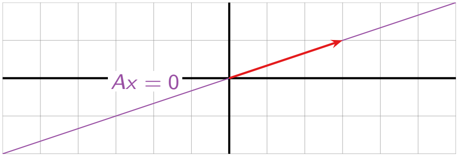
Example 3, figure

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for any x_2 in $\mathbf{R}$. The solution set is $\mathrm{Span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$.



Note: *one* free variable means the solution set is a *line* in $\mathbf{R}^2$ ($2 = \#$ variables $= \#$ columns).

Example 2, with figure

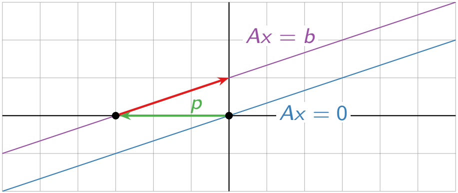
Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ for any x_2 in $\mathbf{R}$.

This is a *translate* of $\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$: it is the parallel line through $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$.



It can be written

$$\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

Parametric vector form

Example 4, explained

Question

What is the solution set of $Ax = 0$, where $A =$

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equations}} \begin{cases} x_1 \qquad - 8x_3 - 7x_4 = 0 \\ \qquad x_2 + 4x_3 + 3x_4 = 0 \end{cases}$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 8x_3 + 7x_4 \\ x_2 = -4x_3 - 3x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}?$$

Answer: $\text{Span}\left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$

[not pictured here]

Note: *two* free variables means the solution set is a *plane* in $\mathbf{R}^4$ ($4 = \#$ variables $= \#$ columns).

Everything is easier when $b = 0$, so we start with this case.

Definition

A system of linear equations of the form $Ax = 0$ is called *homogeneous.*

A homogeneous system *always has the solution* $x = 0$. This is called the *trivial solution.* The nonzero solutions are called **nontrivial.**

Observation

$Ax = 0$ has a *nontrivial* solution

$\iff$ there is a free variable

$\iff$ A has a *column with no pivot*.

The opposite:

Definition

A system of linear equations of the form $Ax = b$ with $b \neq 0$ is called **nonhomogeneous** or **inhomogeneous.**

Poll

Poll

How many solutions can there be to a homogeneous system with more equations than variables?

A. 0

B. 1

C. ∞

The trivial solution is always a solution to a homogeneous system, so answer A is impossible.

This matrix has only one solution to $Ax = 0$:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

This matrix has infinitely many solutions to $Ax = 0$:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Solutions for Homogeneous Systems

Let c be a scalar, u, v be vectors, and A a matrix.

- $A(u+v) = Au + Av$
- $A(cv) = cAv$

See Lay, §1.4, Theorem 5.

For instance, $A(3u - 7v) = 3Au - 7Av$.

Consequence: If u and v are solutions to $Ax = 0$, then so is every vector in Span$\{u, v\}$. Why?

$$\begin{cases} Au = 0 \\ Av = 0 \end{cases} \implies A(c_1u + c_2v) = c_1Au + c_2Av = c_10 + c_20 = 0.$$

(Here 0 means the zero vector.)

Important

The set of **solutions to $Ax = 0$** is **a span**.

Solutions for **Consistent** Nonhomogeneous Systems

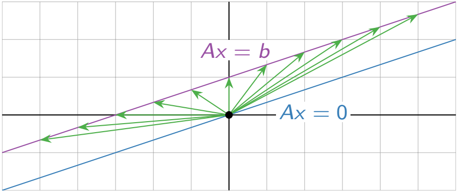
When consistent

The set of **solutions to $Ax = b$**, is **parallel** to a span.

Why? solutions are obtained by taking one **specific** or **particular solution** p to $Ax = b$, and *adding all solutions to $Ax = 0$*.

If $Ap = b$ and $Ax = 0$, then $p + x$ is also a solution to $Ax = b$:

$$A(p + x) = Ap + Ax = b + 0 = b.$$



Note:
Works for *any specific solution p*: it doesn't matter how one found it!

An homogeneous System

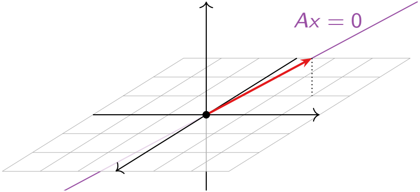
Example 5

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix}?$$

Answer: $\text{Span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}$.



Note: *one* free variable means the solution set is a *line* in $\mathbf{R}^3$ ($3 = \#$ variables $= \#$ columns).

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equations}} \begin{cases} x_1 - 2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 2x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{cases}$$

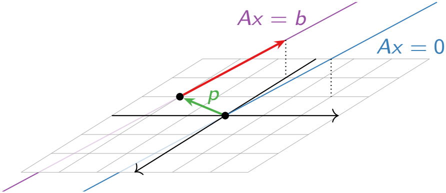
$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -5 \\ -3 \\ -2 \end{pmatrix}?$$

Answer: $\text{Span}\left\{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}.$



The solution set is a *translate* of

$$\text{Span}\left\{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right\}:$$

it is the parallel line through

$$p = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}.$$

A Nonhomogeneous System

Example 6, explained

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -5 \\ -3 \\ -2 \end{pmatrix} ?$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & -5 \\ 2 & -1 & -5 & -3 \\ 1 & 0 & -2 & -2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\text{equations}} \begin{cases} x_1 \qquad - 2x_3 = -2 \\ \qquad x_2 + x_3 = -1 \end{cases}$$

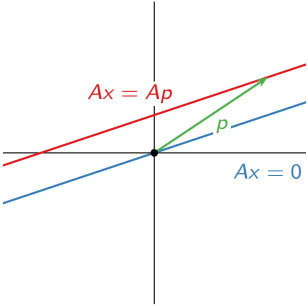
$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 2x_3 - 2 \\ x_2 = -x_3 - 1 \\ x_3 = x_3 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}.$$

Extra: Reverse Engineering

Question

Give a system whose solution set passes through point p and it is *parallel to* the solution set of $Ax = 0$.



1. Set $b = Ap$.
   Entries in p are the weights that produce b *as a linear combination* of columns of A.
2. Now *p is a specific solution* to $Ax = b$,
3. so $Ax = b$ **is the system** we wanted.

Take out: If we describe the solution set of $Ax = 0$, then we can describe the solution set of $Ax = b$ for *all b* in the *Span of columns of A*.