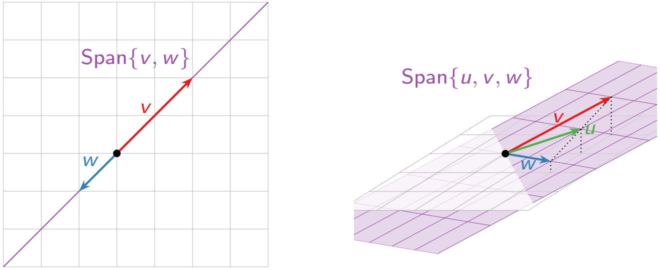
Section 1.7

Linear Independence

Motivation

Sometimes the *span* of a set of vectors *"is smaller"* than you expect from the number of vectors.



This "means" you *don't need so many vectors* to express the same set of vectors.
Notice in each case that one vector in the set is already in the span of the others—so it doesn't make the span bigger.

Today we will formalize this idea in the concept of *linear (in)dependence*.

Linear Independence

Definition

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has **only the trivial solution** $x_1 = x_2 = \cdots = x_p = 0$.

The opposite:

The set $\{v_1, v_2, \ldots, v_p\}$ is *linearly dependent* if there exist numbers $x_1, x_2, \ldots, x_p$, not all equal to zero, such that

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0.$$

This is called a *linear dependence relation*.

Like span, linear (in)dependence is another one of those big vocabulary words that you absolutely need to learn. Much of the rest of the course will be built on these concepts, and you need to know exactly what they mean in order to be able to answer questions on quizzes and exams (and solve real-world problems later on).

Linear Independence

Definition

A set of vectors $\{v_1, v_2, \dots, v_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$. The set $\{v_1, v_2, \dots, v_p\}$ is **linearly dependent** otherwise.

> The notion of linear (in)dependence *applies to a collection of vectors*, not to a single vector, or to one vector in the presence of some others.

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}3\\1\\4\end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

$$x\begin{pmatrix}1\\1\\1\end{pmatrix} + y\begin{pmatrix}1\\-1\\2\end{pmatrix} + z\begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

have a nontrivial solution? How do we solve this kind of vector equation?

$$\begin{pmatrix}1 & 1 & 3\\1 & -1 & 1\\1 & 2 & 4\end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 0\end{pmatrix}$$

So $x = -2z$ and $y = -z$. So the vectors are linearly *de*pendent, and an equation of linear dependence is (taking $z = 1$)

$$-2\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\-1\\2\end{pmatrix} + \begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

have a nontrivial solution?

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 0 & 2 & 4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The trivial solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the unique solution. So the vectors are linearly *in*dependent.

Linear Independence and Matrix Columns

By definition, $\{v_1, v_2, \ldots, v_p\}$ is *linearly independent* if and only if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution. This holds *if and only if* the matrix equation

$$Ax = 0$$

has only the trivial solution, where A is the *matrix with columns* $v_1, v_2, \ldots, v_p$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

This is true if and only if the matrix A has *a pivot in each column*.

Important

- The vectors $v_1, v_2, \ldots, v_p$ are linearly independent if and only if the matrix with columns $v_1, v_2, \ldots, v_p$ has a pivot in each column.
- Solving the matrix equation $Ax = 0$ will either verify that the columns $v_1, v_2, \ldots, v_p$ of A are linearly independent, or will produce a linear dependence relation.

If one of the vectors $\{v_1, v_2, \dots, v_p\}$ is a linear combination of the other ones:

$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$$

Then the vectors are linearly *de*pendent:

$$2v_1 - \frac{1}{2}v_2 - v_3 + 6v_4 = 0.$$

Conversely, if the vectors are linearly dependent

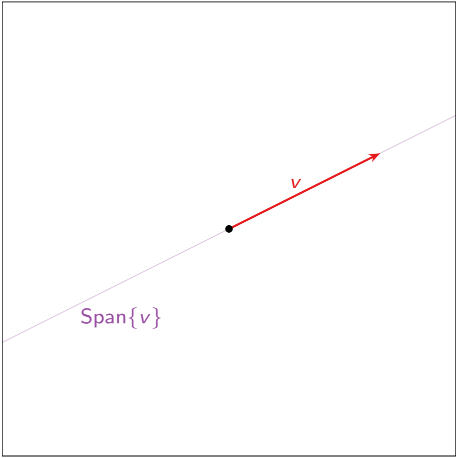
$$2v_1 - \frac{1}{2}v_2 + 6v_4 = 0,$$

then one vector is a linear combination of the other ones:

$$v_2 = 4v_1 + 12v_4.$$

Theorem

A set of vectors $\{v_1, v_2, \dots, v_p\}$ is **linearly dependent** if and only if *one* of the vectors is *in the span of the other* ones.

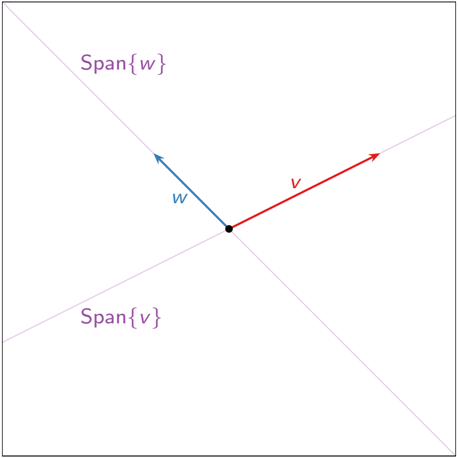


In this picture

One vector $\{v\}$:
Linearly independent **if** $v \neq 0$.

Linear Independence

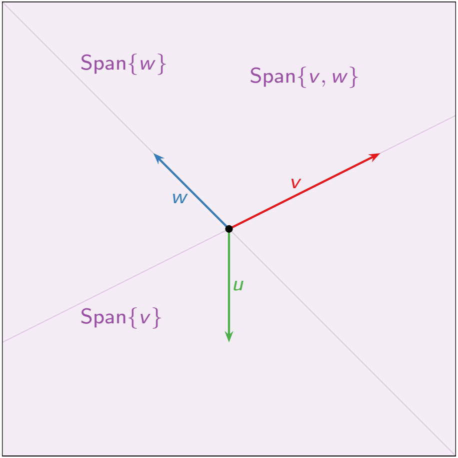
Pictures in $\mathbf{R}^2$



In this picture

One vector $\{v\}$:
Linearly independent **if** $v \neq 0$.

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.



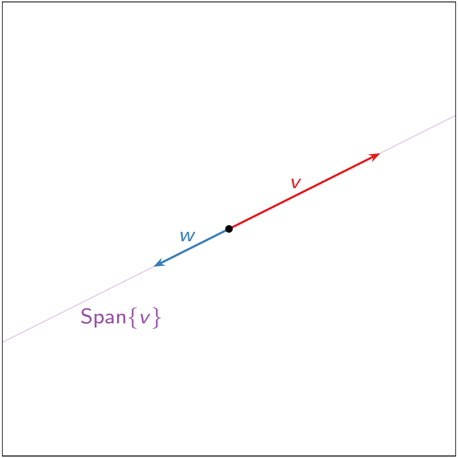
In this picture

One vector $\{v\}$:
Linearly independent **if** $v \neq 0$.

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

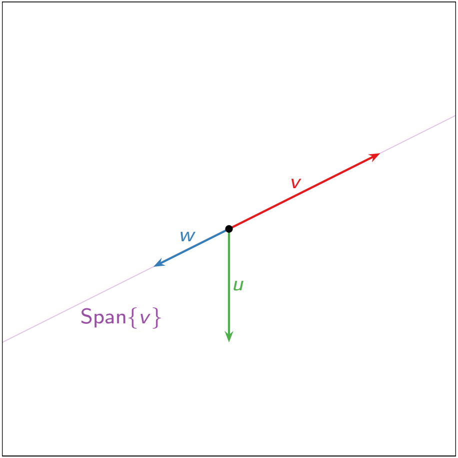
Three vectors $\{v, w, u\}$:
Linearly dependent: u is in $\mathrm{Span}\{v, w\}$.

Also
v is in $\mathrm{Span}\{u, w\}$ and
w is in $\mathrm{Span}\{u, v\}$.



Two collinear vectors $\{v, w\}$:
Linearly dependent: w is in Span$\{v\}$ (and vice-versa).

- *Two vectors* are linearly **dependent** if and only if they are *collinear*.

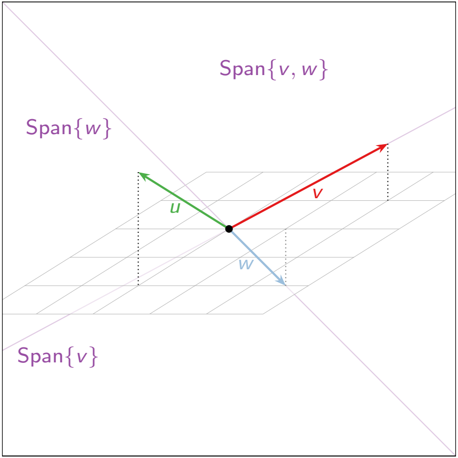


Two collinear vectors $\{v, w\}$:
Linearly dependent: w is in Span$\{v\}$ (and vice-versa).

- *Two vectors* are linearly **dependent** if and only if they are *collinear*.

Three vectors $\{v, w, u\}$:
Linearly dependent: w is in Span$\{v\}$ (and vice-versa).

- If a *set of vectors* is linearly **dependent**, then so is *any larger set* of vectors!



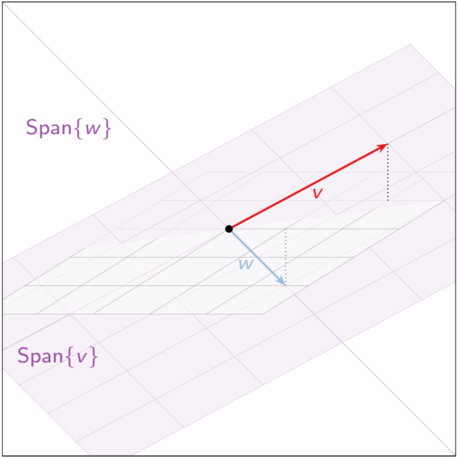
In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, u\}$:
Linearly independent: no one is in the span of the other two.

Linear Independence

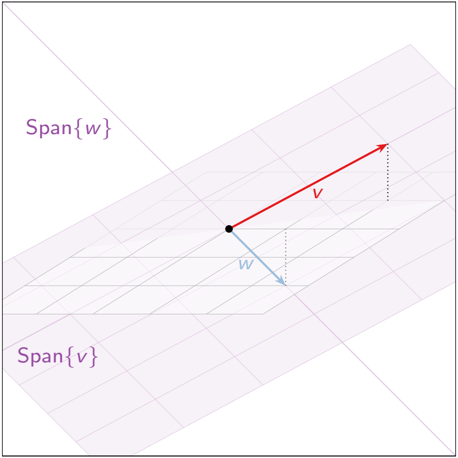
Pictures in $\mathbf{R}^3$



In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, x\}$:
Linearly dependent: x is in $\mathrm{Span}\{v, w\}$.



In this picture

Two vectors $\{v, w\}$:
Linearly independent: neither is in the span of the other.

Three vectors $\{v, w, x\}$:
Linearly dependent: x is in Span$\{v, w\}$.

Which subsets are linearly dependent?

Think about

Are there four vectors u, v, w, x *in* $\mathbf{R}^3$ which are *linearly dependent*, but such that u **is not a linear combination of** v, w, x? If so, draw a picture; if not, give an argument.

Yes: actually the pictures on the previous slides provide such an example.

Linear dependence of $\{v_1, \ldots, v_p\}$ **means some** v_i **is** a linear combination of the others, **not any**.

Linear Dependence

Stronger criterion

Suppose a set of vectors $\{v_1, v_2, \ldots, v_p\}$ is *linearly dependent*.

Take the **largest** j such that v_j *is in the span* of the others.

$$\text{Is } v_j \text{ is in the } \textit{span of } v_1, v_2, \ldots, v_{j-1}?$$

For example, $j = 3$ and

$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$$

Rearrange:

$$v_4 = -\frac{1}{6}\left(2v_1 - \frac{1}{2}v_2 - v_3\right)$$

so v_4 is also in the span of v_1, v_2, v_3, but v_3 was supposed to be the last one that was in the span of the others.

Better Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is **linearly dependent** if and only if there is some j such that v_j *is in* $\mathrm{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

If the vector v_j **is not in** $\text{Span}\{v_1, v_2, \ldots, v_{j-1}\}$,

it means $\text{Span}\{v_1, v_2, \ldots, v_j\}$ *is bigger* than $\text{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

If true for all j

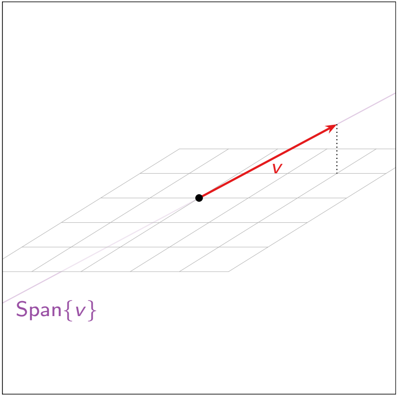
A set of vectors is linearly independent if and only if, every time *you add another vector* to the set, the *span gets bigger*.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is **linearly independent** if and only if, *for every j*, the *span of $v_1, v_2, \ldots, v_j$ is strictly larger* than the span of $v_1, v_2, \ldots, v_{j-1}$.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is **linearly independent** if and only if, *for every* j, the *span of* $v_1, v_2, \ldots, v_j$ *is strictly larger* than the span of $v_1, v_2, \ldots, v_{j-1}$.

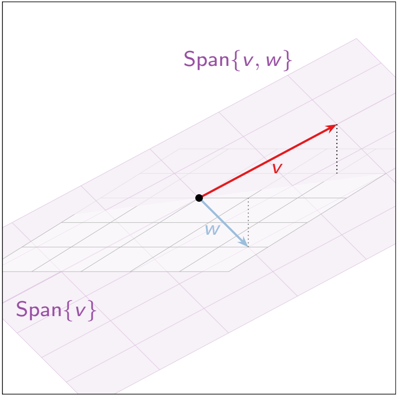


One vector $\{v\}$:

Linearly independent: span got bigger (than $\{0\}$).

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is **linearly independent** if and only if, *for every* j, the *span of* $v_1, v_2, \ldots, v_j$ *is strictly larger* than the span of $v_1, v_2, \ldots, v_{j-1}$.

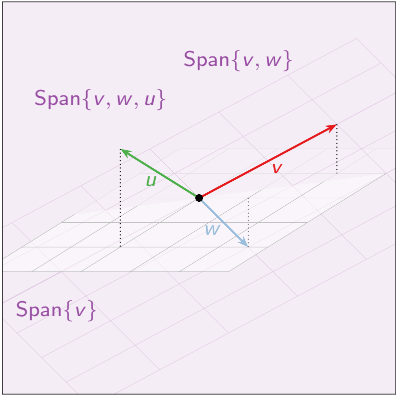


One vector $\{v\}$:
Linearly independent: span got bigger (than $\{0\}$).

Two vectors $\{v, w\}$:
Linearly independent: span got bigger.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is **linearly independent** if and only if, *for every j*, the *span of $v_1, v_2, \ldots, v_j$ is strictly larger* than the span of $v_1, v_2, \ldots, v_{j-1}$.



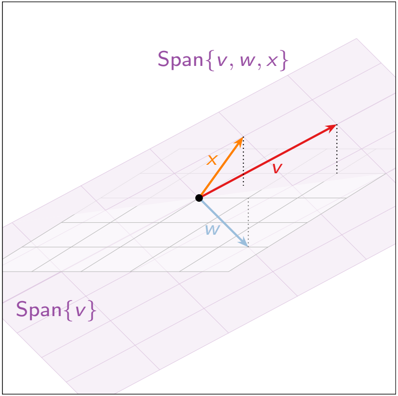
One vector $\{v\}$:
Linearly independent: span got bigger (than $\{0\}$).

Two vectors $\{v, w\}$:
Linearly independent: span got bigger.

Three vectors $\{v, w, u\}$:
Linearly independent: span got bigger.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is **linearly independent** if and only if, *for every j*, the *span of $v_1, v_2, \ldots, v_j$ is strictly larger* than the span of $v_1, v_2, \ldots, v_{j-1}$.



One vector $\{v\}$:
Linearly independent: span got bigger (than $\{0\}$).

Two vectors $\{v, w\}$:
Linearly independent: span got bigger.

Three vectors $\{v, w, x\}$:
Linearly dependent: span didn't get bigger.

Extra: Linear Independence

Two more facts

Fact 1: Say $v_1, v_2, \ldots, v_n$ are in $\mathbf{R}^m$. *If $n > m$* then $\{v_1, v_2, \ldots, v_n\}$ is *linearly dependent*: the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

cannot have a pivot in each column (it is too wide).

This says you can't have 4 linearly independent vectors in $\mathbf{R}^3$, for instance.

A wide matrix can't have linearly independent columns.

Fact 2: *If one* of $v_1, v_2, \ldots, v_n$ *is zero*, then $\{v_1, v_2, \ldots, v_n\}$ is *linearly dependent*. For instance, if $v_1 = 0$, then

$$1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + \cdots + 0 \cdot v_n = 0$$

is a linear dependence relation.

A set containing the zero vector is linearly dependent.