Section 1.9

The Matrix of a Linear Transformation

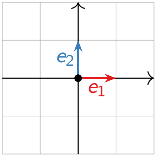
Unit Coordinate Vectors

Definition
The **unit coordinate vectors** in $\mathbf{R}^n$ are

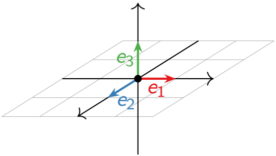
This is what $e_1, e_2, \ldots$ mean, *for the rest of the class.*

$$\boxed{e_1 = \begin{pmatrix}1\\0\\\vdots\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\\vdots\\0\\0\end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix}0\\0\\\vdots\\1\\0\end{pmatrix}, \quad e_n = \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}.}$$

in $\mathbf{R}^2$



in $\mathbf{R}^3$



Important: if A is an $m \times n$ matrix with *columns* $v_1, v_2, \ldots, v_n$, then $Ae_i = v_i$ for $i = 1, 2, \ldots, n$: the transformation $T(x) = Ax$ sends e_i *to vector* v_i.

$$\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\5\\8\end{pmatrix}$$

Recap: Linear Transformations

Recall: If A is a matrix, u, v are vectors, and c is a scalar, then

$$A(u+v) = Au + Av \qquad A(cv) = cAv.$$

So if $T(x) = Ax$ is a matrix transformation then,

$$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cu) = cT(u)$$

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if it satisfies the above equations for *all vectors* u, v in $\mathbf{R}^n$ and *all scalars* c.

In other words, T **"respects" addition and scalar multiplication**.

More generally, (in engineering this is called **superposition**)

$$T\big(c_1v_1 + c_2v_2 + \cdots + c_nv_n\big) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n).$$

So that *unit coordinate vectors* determine where all vectors in $\mathbf{R}^n$ get mapped to in $\mathbf{R}^m$.

Linear Transformations are Matrix Transformations

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a **linear transformation**. Let

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

This is an $m \times n$ matrix, and T is the matrix transformation for A: $T(x) = Ax$.

The matrix A is called the *standard matrix for T*.

Take-Away

A linear transformation *may not be given* a priori
as a matrix transformation
but **linear** transformations are **the same as matrix** transformations.

Dictionary

Linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ $\rightsquigarrow$ $m \times n$ matrix $A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}$

$T(x) = Ax$

Linear Transformations: Dilation

Before, we defined a **dilation** transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$. What is its standard matrix?

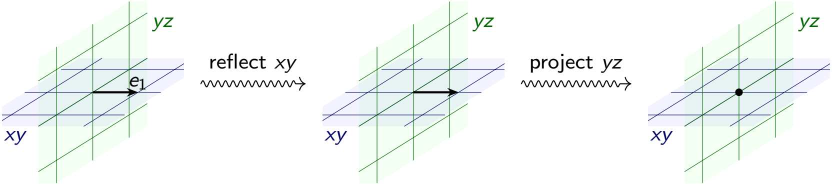
$$\left.\begin{aligned} T(e_1) &= 1.5e_1 = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} \\ T(e_2) &= 1.5e_2 = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix} \end{aligned}\right\} \implies A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}.$$

Check:

$$\begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.5x \\ 1.5y \end{pmatrix} = 1.5\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix}.$$

Question

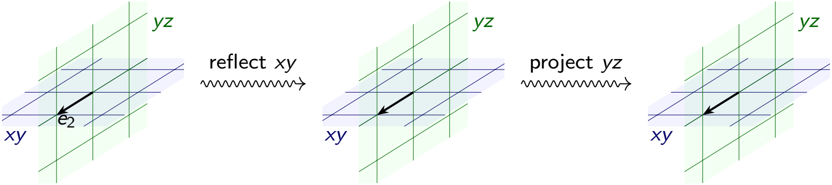
What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?



$$T(e_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Question

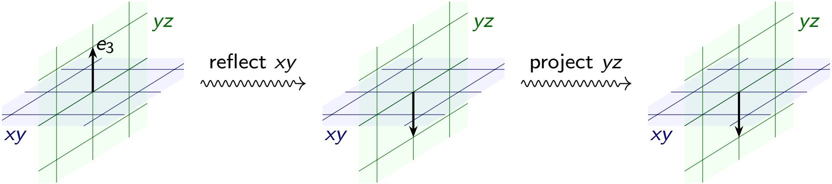
What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?



$$T(e_2) = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?



$$T(e_3) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

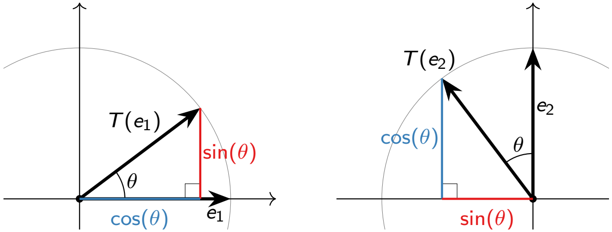
$$\left.\begin{aligned} T(e_1) &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ T(e_2) &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ T(e_1) &= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \end{aligned}\right\} \Longrightarrow A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Linear Transformations: Rotation

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$T(x) = x \text{ rotated counterclockwise by an angle } \theta?$$



$$\left.\begin{array}{l} T(e_1) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \\[1em] T(e_2) = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \end{array}\right\} \implies A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \qquad \left(\begin{array}{c} \theta = 90^\circ \implies \\ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \text{(from before)} \end{array} \right)$$

Check linearity . . .

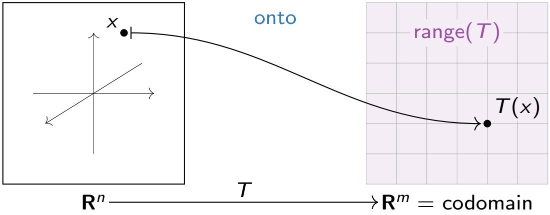
Other Geometric Transformations

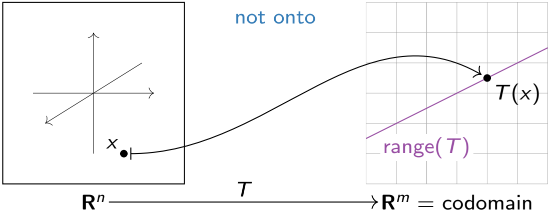
There is a long list of geometric transformations of $\mathbf{R}^2$ in §1.9 of Lay. (Reflections over the diagonal, contractions and expansions along different axes, shears, projections, ...) Please look them over.

Onto Transformations

Definition

A transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is **onto** (or **surjective**) if the *range of T* is equal to $\mathbf{R}^m$ (its codomain). In other words, *each b in $\mathbf{R}^m$ is the image of at least one x* in $\mathbf{R}^n$: every possible output has an input. Note that *not* onto means there is some b in $\mathbf{R}^m$ which is not the image of any x in $\mathbf{R}^n$.





Characterization of Onto Transformations

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A. *Then the following are equivalent:*

- **T is onto**
- $T(x) = b$ has a solution for every b in $\mathbf{R}^m$
- $Ax = b$ is *consistent for every b* in $\mathbf{R}^m$
- A has a pivot in every row
- **The columns of A span $\mathbf{R}^m$**

Question

If $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is onto, what can we say about the relative sizes of n and m?

Answer: A must have *at least as many columns as rows* $(m \leq n)$ to have a pivot in every row.

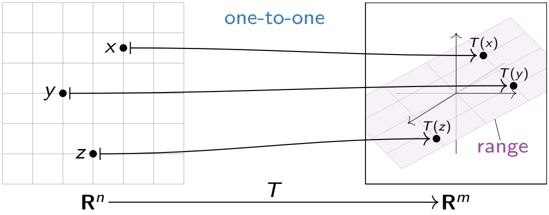
$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{pmatrix}$$

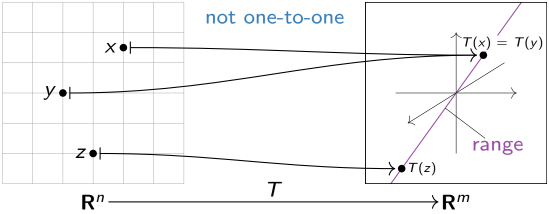
For instance,
$\mathbf{R}^2$ is "too small"
to map *onto* $\mathbf{R}^3$.

One-to-one Transformations

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **one-to-one** (or **into**, or **injective**) if *different vectors* in $\mathbf{R}^n$ *map to different vectors* in $\mathbf{R}^m$. In other words, each b in $\mathbf{R}^m$ is the image of *at most one* x in $\mathbf{R}^n$: different inputs have different outputs. Note that *not* one-to-one means different vectors in $\mathbf{R}^n$ have the same image.





Characterization of One-to-One Transformations

Theorem

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A. *Then the following are equivalent*:

- **T is one-to-one**
- $T(x) = b$ has one or zero solutions for every b in $\mathbf{R}^m$
- $Ax = b$ has a *unique solution or is inconsistent* for every b in $\mathbf{R}^m$
- $Ax = 0$ has a unique solution
- A has a pivot in every column.
- The **columns of A are linearly independent**

Question

If $T : \mathbf{R}^n \to \mathbf{R}^m$ is one-to-one, what can we say about the relative sizes of n and m?

Answer: A must have *at least as many rows as columns* ($n \le m$) to have a pivot in every column.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

For instance, $\mathbf{R}^4$ is "too big" to map *into* $\mathbf{R}^2$.

Why is a linear transformation a matrix transformation?

Suppose for simplicity that $T\colon \mathbf{R}^3 \to \mathbf{R}^2$.

$$\begin{aligned}
T\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= T\left(x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \\
&= T\big(xe_1 + ye_2 + ze_3\big) \\
&= xT(e_1) + yT(e_2) + zT(e_3) \\
&= \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= A\begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\end{aligned}$$