# Section 2.3

## Characterization of Invertible Matrices

# Invertible Transformations

## Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is **invertible** if there exists $U\colon \mathbf{R}^n \to \mathbf{R}^n$ such that for all $x$ in $\mathbf{R}^n$

$$T \circ U(x) = x \quad \text{and} \quad U \circ T(x) = x.$$

In this case we say $U$ is the **inverse** of $T$, and we **write** $U = T^{-1}$.

In other words, $T(U(x)) = x$, so $T$ *"undoes"* $U$, and likewise $U$ "undoes" $T$.

> **Fact**
>
> A transformation $T$ is invertible if and only if *it is both one-to-one and onto*.

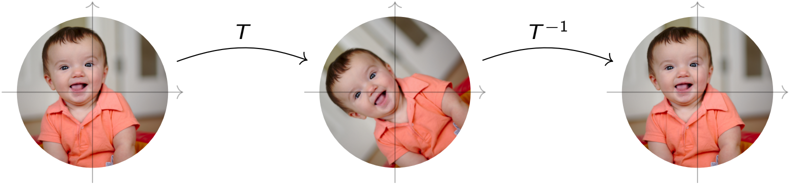
This means *for every $y$* in $\mathbf{R}^n$, *there is a unique $x$* in $\mathbf{R}^n$ such that $T(x) = y$.

Therefore we can define $T^{-1}(y) = x$.

# Invertible Transformations

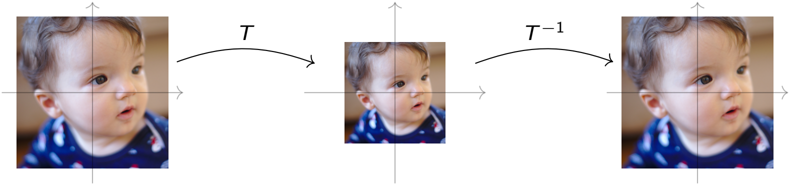
## Examples

Let $T =$ **counterclockwise** rotation in the plane by $45^\circ$. What is $T^{-1}$?




$T^{-1}$ is *clockwise* rotation by $45^\circ$.

Let $T =$ **shrinking** by a factor of $2/3$ in the plane. What is $T^{-1}$?




$T^{-1}$ is *stretching* by $3/2$.

# Invertible Linear Transformations

Let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be **an invertible linear transformation** with matrix $A$.

Let $B$ be the matrix for $T^{-1}$. We know $T \circ T^{-1}$ has matrix $AB$, so for all $x$,

$$ABx = T \circ T^{-1}(x) = x.$$

*Hence* $AB = I_n$, that is $B = A^{-1}$ (This is why we define matrix inverses).

**Fact**

If *$T$ is an invertible* linear transformation with *matrix $A$*, then $T^{-1}$ is an invertible linear transformation with **matrix** $A^{-1}$.

**Non-invertibility**: E.g. let $T =$ *projection* onto the $x$-axis. What is $T^{-1}$? *It is not invertible*: you can't undo it.

It's corresponding matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is not invertible!

# Invertible transformations

## Example 1

Let $T =$ *shrinking by a factor of* $2/3$ in the plane. Its matrix is

$$A = \begin{pmatrix} 2/3 & 0 \\ 0 & 2/3 \end{pmatrix}$$

Then $T^{-1} =$ *stretching by* $3/2$. Its matrix is

$$B = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$$

Check:

$$AB = \begin{pmatrix} 2/3 & 0 \\ 0 & 2/3 \end{pmatrix} \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark$$

The matrix corresponding to $T \circ T^{-1}$ is $AB$, which satisfies $(AB)x = x$

Note: the matrix corresponding to $T^{-1} \circ T$ is $BA$, also satisfies $(BA)x = x$

# Invertible transformations

## Example 2

Let $T$ = counterclockwise *rotation* in the plane *by* $45^\circ$. Its matrix is

$$A = \begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then $T^{-1}$ = counterclockwise *rotation by* $-45^\circ$. Its matrix is

$$B = \begin{pmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & \cos(-45^\circ) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Check:

$$AB = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BA = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ✓$$

The matrix corresponding to $T \circ T^{-1}$ is $AB$, which satisfies $(AB)x = x$ Note:

the matrix corresponding to $T^{-1} \circ T$ is $BA$, also satisfies $(BA)x = x$

# The Really Big Theorem for Square Matrices of Math 1553

## The Invertible Matrix Theorem

Let $A$ **be an** $n \times n$ **matrix**, and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be defined by $T(x) = Ax$.
*The following statements are equivalent*.

1. $A$ is invertible.
2. $T$ is invertible.
3. $T$ is one-to-one.
4. $T$ is onto.
5. $A$ has a left inverse (there exists $B$ such that $BA = I_n$).
6. $A$ has a right inverse (there exists $B$ such that $AB = I_n$).
7. $A^T$ is invertible.
8. $A$ is row equivalent to $I_n$.
9. $A$ has $n$ pivots (one on each column and row).
10. The columns of $A$ are linearly independent.
11. $Ax = 0$ has only the trivial solution.
12. The columns of $A$ span $\mathbf{R}^n$.
13. $Ax = b$ is consistent for all $b$ in $\mathbf{R}^n$.

you really have to understand these

# Approach to The Invertible Matrix Theorem

As with all Equivalence theorems:

- For **invertible matrices**: all statements of the Invertible Matrix Theorem are true.
- For **non-invertible matrices**: *all statements* of the Invertible Matrix Theorem *are false*.

**Tackle the assertions!**

**You know enough** at this point to be able to *reduce all* of the statements *to assertions about the pivots* of a square matrix.

Strong recommendation: If you want to understand invertible matrices, go through all of the conditions of the IMT and *try to figure out on your own* why they're all equivalent.