Section 2.8

Subspaces of $\mathbf{R}^n$

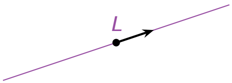
Subspaces: Motivation and examples

Example

The subset $\{0\}$: this subspace contains only one vector.

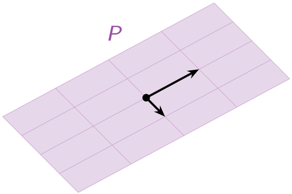
Example

A line L *through the origin*:
this contains the span of any vector in L.



Example

A plane P *through the origin*:
this contains the span of any two vectors in P.



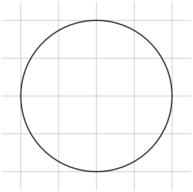
Example

All of $\mathbf{R}^n$:
this contains 0, and is closed under addition and scalar multiplication.

- The **span was our first example** of subspace: $Span\{v_1, \ldots, v_p\}$.
- But in general, subspaces are not defined by *'the generating vectors'*

Subsets vs. Subspaces

A *subset* of $\mathbf{R}^n$ is *any collection* of vectors whatsoever.
All non-examples of subspaces will be still subsets of $\mathbf{R}^n$.



Subset: *yes*
Subspace: *no*

A **subspace** is a special kind of subset, which satisfies *three defining properties:*

1. "not empty"
2. "closed under addition"
3. "closed under $\times$ scalars"

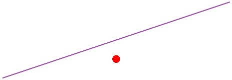
Non-Examples

Color code

Purple: wanna-be 'subspaces'
Red vectors: **would have to be in** the subset too.

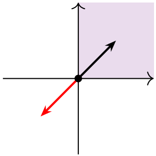
Non-Example

Any set that *doesn't contain the origin*
Fails condition (1).



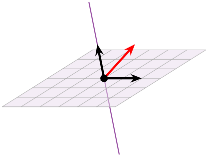
Non-Example

The first quadrant in $\mathbf{R}^2$.
Fails close under $\times$ scalar only.



Non-Example

A line union a plane in $\mathbf{R}^3$.
Fails close under addition only.



The Definition of Subspace

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. *The zero* vector *is in* V. "not empty"
2. If u *and* v are in V, *then* $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, *then* cu is in V. "closed under $\times$ scalars"

Consequences of definition:

- By (3), *if* v is in V, then *so is the line through* v.
- By (2),(3), *if* u, v are in V, then *so is* $xu + yv$, for all $x, y \in \mathbf{R}$.

A subspace V *contains the span* of any set *of vectors in* V.

Spans are Subspaces

Theorem

Any $\text{Span}\{v_1, v_2, \ldots, v_n\}$ is a subspace.

Check:

1. $0 = 0v_1 + 0v_2 + \cdots + 0v_n$ is in the span.
2. If, say, $u = 3v_1 + 4v_2$ and $v = -v_1 - 2v_2$, then

   $$u + v = 3v_1 + 4v_2 - v_1 - 2v_2 = 2v_1 + 2v_2$$

   is also in the span.
3. Similarly, if u is in the span, then so is cu for any scalar c.

Definition

If $V = \text{Span}\{v_1, v_2, \ldots, v_n\}$, we say that V is the subspace **generated by** or **spanned by** the vectors $v_1, v_2, \ldots, v_n$.

!!!

Every span is a subspace but also every subspace is a span.

How would you *find the generating vectors*?

Poll

Poll

Why is the first quadrant of $\mathbf{R}^2$ not a subspace? Which property(ies) does it fail?

$$Q = \left\{ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} : a_1 \geq 0, a_2 \geq 0 \right\}$$

1. The *zero vector is contained* in the first quadrant: ✔
2. It is *closed under addition*: $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in Q$ then

$$a_1 + b_1 \geq 0 \quad a_2 + b_2 \geq 0 \Rightarrow \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} \in Q$$ ✔

3. It is **not closed under** $\times$ scalar: let $a_1, a_2 > 0$ and $c < 0$ then

$$c \cdot a_1 < 0 \quad c \cdot a_2 < 0 \Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in Q, \quad c \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \notin Q$$ ✘

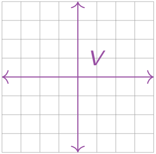
Subspaces

Verification

Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; ab = 0 \right\}$. Let's check if V is a subspace or not.

1. Does V contain the zero vector? $\binom{a}{b} = \binom{0}{0} \implies ab = 0$

3. Is V closed under scalar multiplication?
   - Let $\binom{a}{b}$ be in V. (a *and* b *such that* $ab = 0$).
     Let c be a scalar.
   - *Is* $c\binom{a}{b} = \binom{ca}{cb}$ in V?
     Yes, since $(ca)(cb) = c^2(ab) = c^2(0) = 0$



2. Is V closed under addition?
   - Let $\binom{a}{b}$ and $\binom{a'}{b'}$ be in V. ($ab = 0$ *and* $a'b' = 0$).
   - *Is* $\binom{a}{b} + \binom{a'}{b'} = \binom{a+a'}{b+b'}$ in V?
   - Need to have $(a + a')(b + b') = 0$ **always**.
     However, for $a = b' = 0$ and $a' = b = 1$ this is not true.
   - $\binom{1}{0}$, $\binom{0}{1}$ are in V, but $\binom{1}{1}$ is not in V, ($1 \cdot 1 \neq 0$).

We conclude that V is *not* a subspace. A picture is above.

Basis and dimension of a Subspace

Let V be a subspace of $\mathbf{R}^n$ and $\{v_1, v_2, \ldots, v_m\}$ in V linearly independent. So that every time you *'add one'* of these vectors, the *span gets bigger*.

What if $Span\{v_1, \ldots, v_m\} = V$?

Then any smaller set can't span V. If we *remove any vector*, the span gets *smaller*:

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in V such that:

1. $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is *linearly independent*.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Important

A subspace has many *different bases*, but they all have the *same number* of vectors (see the exercises in §2.9).

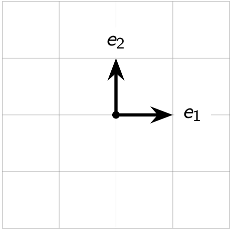
Bases of $\mathbf{R}^2$

Question

What is a basis for $\mathbf{R}^2$?

We need two vectors that *span* $\mathbf{R}^2$ and are *linearly independent*. $\{e_1, e_2\}$ is one basis.

1. They span: $\binom{a}{b} = ae_1 + be_2$.
2. They are linearly independent.

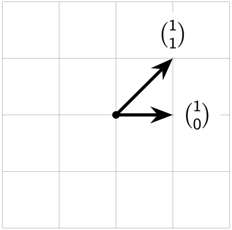


Question

What is another basis for $\mathbf{R}^2$?

Any two *nonzero vectors* that are *not collinear*.
$\left\{ \binom{1}{0}, \binom{1}{1} \right\}$ is also a basis.

1. They span: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has a pivot in every row.
2. They are linearly independent: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has a pivot in every column.



Bases of $\mathbf{R}^n$

The unit coordinate vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

are a basis for $\mathbf{R}^n$. The identity matrix has columns $e_1, e_2, \ldots, e_n$.

1. They span: I_n has a pivot in every row.
2. They are linearly independent: I_n has a pivot in every column.

In general:

$\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^n$ if and only if the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

has a pivot in every row and every column, i.e. if *A is invertible*.

Example

Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x + 3y + z = 0 \right\} \qquad \mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}.$$

Verify that $\mathcal{B}$ *is a basis* for V.

0. In V: both vectors satisfy the equation, so are in V

$$-3 + 3(1) + 0 = 0 \quad \text{and} \quad 0 + 3(1) + (-3) = 0.$$

1. Span: If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in V, then $y = -\frac{1}{3}(x + z)$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{x}{3} \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} - \frac{z}{3} \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}.$$

2. Linearly independent:

$$c_1 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = 0 \implies \begin{pmatrix} -3c_1 \\ c_1 + c_2 \\ -3c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies c_1 = c_2 = 0.$$

Subspaces of a transformation

An $m \times n$ matrix A naturally gives rise to *two* subspaces.

Definition

The *column space* of A is the subspace of $\mathbf{R}^m$ spanned by the columns of A. It is *written* Col A.

The **null space** of A is a subspace of $\mathbf{R}^n$ containing the set of all solutions of the homogeneous equation $Ax = 0$:

$$\text{Nul}\, A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$

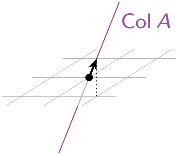
Some remarks:

- The column space is the range (as opposed to the codomain) of the transformation $T(x) = Ax$.
- The column space is defined as a span, so we know it is a subspace.
- For the null space is easier to verify it is a subspace than find its generators. (This is one reason subspaces are so useful.)

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Let's compute the *column space*:

$$\text{Col}\,A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$



This *is a line in* $\mathbf{R}^3$.

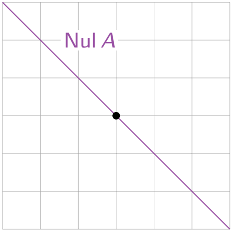
Let's compute the **null space**:

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \\ x+y \end{pmatrix}.$$

This zero if and only if $x = -y$. So

$$\text{Nul}\,A = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \mid y = -x \right\}.$$

This defines **a line in $\mathbf{R}^2$**:



Verify: The null space is a subspace and a span

Check that the null space is a subspace:

1. 0 is in $\operatorname{Nul} A$ because $A0 = 0$.
2. If u and v are in $\operatorname{Nul} A$, then $Au = 0$ and $Av = 0$. Hence
$$A(u + v) = Au + Av = 0,$$
so $u + v$ is in $\operatorname{Nul} A$.
3. If u is in $\operatorname{Nul} A$, then $Au = 0$. For any scalar c, $A(cu) = cAu = 0$. So cu is in $\operatorname{Nul} A$.

Question

How to find vectors which span the null space? Answer: Parametric vector form!

We know that the *solution set to $Ax = 0$* has a parametric form that *looks like*

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

if, *say*, x_3 and x_4 are the *free variables*. So

$$\operatorname{Nul} A = \operatorname{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Refer back to the slides for §1.5 (Solution Sets).

Find Null Space as a Span

Example

Find vector(s) that span the null space of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

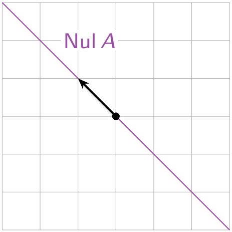
The *reduced row echelon* form is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

This *gives the equation $x + y = 0$*, or

$$\begin{aligned} x &= -y \\ y &= y \end{aligned} \quad \xrightarrow{\text{parametric vector form}} \quad \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The null space is

$$\operatorname{Nul} A = \operatorname{Span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$



Basis for Nul A

Fact

The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for Nul A.

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{parametric vector form}} \quad x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis of Nul } A} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

1. The vectors span Nul A by construction (every solution to $Ax = 0$ has this form).
2. Can you see why they are linearly independent? (Look at the last two rows.)

Basis for Col A

Fact

The *pivot columns* of A always form a basis for Col A.

Warning: It is the pivot columns of the *original matrix* A, **not the row-reduced** form. (Row reduction changes the column space.)

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis ⟵ pivot columns in rref

So a basis for Col A is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}.$$

Why? End of §2.8, or ask in office hours.

How do you check if a subset is a subspace?

- Is it a span?
- Is it all of $\mathbf{R}^n$ or the zero subspace $\{0\}$?

Can it be written as

- a span?
- the column space of a matrix?
- the null space of a matrix?
- a type of subspace that we'll learn about later (eigenspaces, ...)?

If so, then it's automatically a subspace.

If all else fails:

- Can you *verify directly* that it satisfies the *three defining properties*?

Subspaces of a transformation

Recall: a **basis** of a subspace V is a set of vectors that

- *spans* V and
- is *linearly independent*.

Let A be an $m \times n$ matrix.

- The *column space* of A is the subspace of $\mathbf{R}^m$ spanned by the columns of A. It is *written* Col A.
- The **null space** of A is a subspace of $\mathbf{R}^n$ containing the set of all solutions of the homogeneous equation $Ax = 0$:

$$\text{Nul}\, A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$

Basis for Nul A

Fact

The *vectors in* the parametric vector form of the *general solution to $Ax = 0$* always form a **basis for** Nul A.

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$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{parametric vector form}} \quad x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis of Nul } A} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

1. Every solution to $Ax = 0$ has this form. So the *vectors span* Nul A by construction.
2. Look at the *last two rows of the basis*. Can you see why *they are linearly independent*?

Basis for Col A

Fact

The *pivot columns* of A always form a basis for Col A.

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pivot columns = basis ⟵ pivot columns in rref

So a basis for Col A is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}.$$

Why? End of §2.8, or ask in office hours.