# Section 2.9

## Dimension and Rank

# The Rank Theorem

Recall:

- The **dimension** of a subspace $V$ is the number of vectors in a *basis for $V$*.
- A **basis for the column space** of a matrix $A$ is given by the *pivot columns*.
- A **basis for the null space** of $A$ is given by the vectors attached to the *free variables* in the parametric vector form.

### Definition

The **rank** of a matrix $A$, *written* $\operatorname{rank} A$, is the *dimension of the range* of $T(x) = Ax$ (dimension of $\operatorname{Col} A$).

Observe:

$$\begin{aligned} \operatorname{rank} A = \dim \operatorname{Col} A &= \text{the number of columns with pivots} \\ \dim \operatorname{Nul} A &= \text{the number of free variables} \\ &= \text{the number of columns without pivots.} \end{aligned}$$

### Rank Theorem

If $A$ is an $m \times n$ matrix, then

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n = \text{the number of columns of } A.$$

# The Rank Theorem

## Example

$$A = \begin{pmatrix} \boxed{\begin{matrix} 1 \\ -2 \\ 2 \end{matrix}} & \boxed{\begin{matrix} 2 \\ -3 \\ 4 \end{matrix}} & \begin{matrix} 0 \\ 4 \\ 0 \end{matrix} & \begin{matrix} -1 \\ 5 \\ -2 \end{matrix} \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} & \boxed{\begin{matrix} -8 \\ 4 \\ 0 \end{matrix}} & \boxed{\begin{matrix} -7 \\ 3 \\ 0 \end{matrix}} \end{pmatrix}$$

basis of Col $A$

free variables

A basis for Col $A$ is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\},$$

so $\operatorname{rank} A = \dim \operatorname{Col} A = 2$.

Since there are two free variables $x_3, x_4$, the parametric vector form for the solutions to $Ax = 0$ is

$$x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis for Nul } A} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Thus $\dim \operatorname{Nul} A = 2$.

The *Rank Theorem* says $2 + 2 = 4$.

# The Basis Theorem

## Basis Theorem

Let $V$ be a **subspace of dimension** $m$. Then:

- Any *$m$ linearly independent* vectors in $V$ form *a basis* for $V$.
- Any *$m$ vectors that span* $V$ form *a basis* for $V$.

**Upshot**

If you *already* know that $\dim V = m$, and you have $m$ vectors $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ in $V$, then *check only one* of

1. $\mathcal{B}$ is linearly independent, or
2. $\mathcal{B}$ spans $V$

in order **for $\mathcal{B}$ to be a basis**.

# Poll

**Poll**

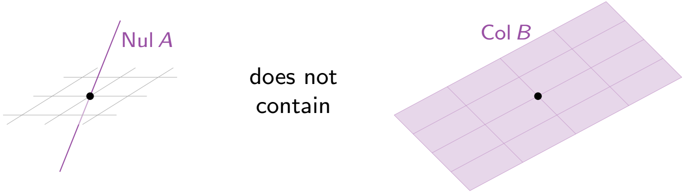
Let $A$ and $B$ be $3 \times 3$ matrices. Suppose that $\operatorname{rank}(A) = 2$ and $\operatorname{rank}(B) = 2$. Is it **possible that $AB = 0$**?

Our information, by the rank theorem:
$\operatorname{rank}(A) = 2$ and $\dim \operatorname{Nul} A = 1$, also
$\operatorname{rank}(B) = \dim \operatorname{Col} B = 2$ and $\dim \operatorname{Nul} B = 1$.

If $AB = 0$, then $ABx = 0$ for every $x$ in $\mathbf{R}^3$.

This means $A(Bx) = 0$ for all $x \in \mathbf{R}^3$. Every vector $Bx$ is in $\operatorname{Nul} A$.

Then the range of $T(x) = Bx$ (same as $\operatorname{Col} B$) is contained in $\operatorname{Nul} A$.
But a 1-dimensional space can't contain a 2-dimensional space.



# Bases as Coordinate Systems

The unit coordinate vectors $e_1, e_2, \ldots, e_n$ *form a basis for* $\mathbf{R}^n$. Any vector is a *unique linear combination* of the $e_i$:

$$v = \begin{pmatrix} 3 \\ 5 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3e_1 + 5e_2.$$

Note that the coordinates of $v$ are exactly the coefficients of $e_1, e_2, e_3$.

Going backwards: for any basis $\mathcal{B}$, we *interpret the coefficients* of a linear combination as **coordinates with respect to $\mathcal{B}$**.

### Definition

Let $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ be a basis of a subspace $V \subset \mathbf{R}^p$ (so $m \leq p$). The coordinates of $x$ with respect to $\mathcal{B}$ are the coefficients $c_1, c_2, \ldots, c_m$ of the *unique linear combination* $x = c_1v_1 + c_2v_2 + \cdots + c_mv_m$.

The **$\mathcal{B}$-coordinate vector of $x$** is the vector

$$[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \quad \text{in } \mathbf{R}^m.$$

# Bases as Coordinate Systems

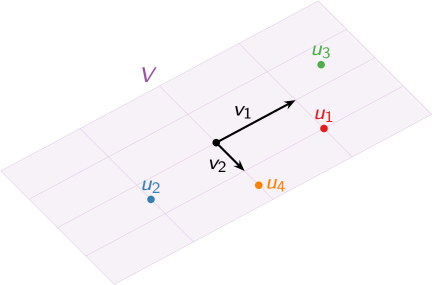
## Picture

Let

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

These form *a basis $\mathcal{B}$* for the plane

$$V = \text{Span}\{v_1, v_2\} \text{in } \mathbf{R}^4.$$



Question: Estimate the *$\mathcal{B}$-coordinates* of these vectors:

$$[u_1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad [u_2]_{\mathcal{B}} = \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix} \qquad [u_3]_{\mathcal{B}} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} \qquad [u_4]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix}$$

### Remark

Make sense of $V$ as two-dim: Choose a basis $\mathcal{B}$ and use $\mathcal{B}$-coordinates.
Careful: The coordinates give *only the coefficients* of a linear combination *using such basis vectors*.

# Bases as Coordinate Systems

## Example 1

Let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathcal{B} = \{v_1, v_2\}$, $V = \text{Span}\{v_1, v_2\}$.

Verify that $\mathcal{B}$ is a basis:
*Span:* by definition $V = \text{Span}\{v_1, v_2\}$.
*Linearly independent:* because they are not multiples of each other.

Question: If $[x]_{\mathcal{B}} = \binom{5}{2}$, then what is $x$?

$$[x]_{\mathcal{B}} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad \text{means} \quad x = 5v_1 + 2v_2 = 5\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 7 \end{pmatrix}.$$

Question: Find the $\mathcal{B}$-coordinates of $x = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}$.

We have to solve the vector equation $x = c_1 v_1 + c_2 v_2$ in the unknowns $c_1, c_2$.

$$\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 1 & 5 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

So $c_1 = 2$ and $c_2 = 3$, so $x = 2v_1 + 3v_2$ and $[x]_{\mathcal{B}} = \binom{2}{3}$.

# Bases as Coordinate Systems

## Example 2

Let $v_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}, \quad V = \mathrm{Span}\{v_1, v_2, v_3\}$.

Question: Find a basis for $V$.
$V$ is the column span of the matrix

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & 8 \\ 2 & 1 & 6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis for the column span is formed by the pivot columns: $\mathcal{B} = \{v_1, v_2\}$.

Question: Find the $\mathcal{B}$-coordinates of $x = \begin{pmatrix} 4 \\ 11 \\ 8 \end{pmatrix}$.

We have to solve $x = c_1 v_1 + c_2 v_2$.

$$\left(\begin{array}{cc|c} 2 & -1 & 4 \\ 3 & 1 & 11 \\ 2 & 1 & 8 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

So $x = 3v_1 + 2v_2$ and $[x]_{\mathcal{B}} = \binom{3}{2}$.

# The Invertible Matrix Theorem

## Addenda

Using the *Rank Theorem* and the *Basis Theorem*, we have new interpretations of the **meaning of invertibility**.

### The Invertible Matrix Theorem

Let $A$ be an $n \times n$ matrix, and let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. *The following statements are equivalent*.

1. **$A$ is invertible**.
2. $T$ is invertible.
3. $A$ is row equivalent to $I_n$.
4. $A$ has $n$ pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of $A$ are linearly independent.
7. $T$ is one-to-one.
8. $Ax = b$ is consistent for all $b$ in $\mathbf{R}^n$.
9. The columns of $A$ span $\mathbf{R}^n$.
10. $T$ is onto.
11. $A$ has a left inverse (there exists $B$ such that $BA = I_n$).
12. $A$ has a right inverse (there exists $B$ such that $AB = I_n$).
13. $A^T$ is invertible.
14. The columns of $A$ form a basis for $\mathbf{R}^n$.
15. $\operatorname{Col} A = \mathbf{R}^n$.
16. $\dim \operatorname{Col} A = n$.
17. $\operatorname{rank} A = n$.
18. $\operatorname{Nul} A = \{0\}$.
19. $\dim \operatorname{Nul} A = 0$.

# Bases as Coordinate Systems

## Summary

If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ is a basis for a subspace $V$ and $x$ is in $V$, then

$$[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad \text{means} \quad x = c_1v_1 + c_2v_2 + \cdots + c_mv_m.$$

Finding the $\mathcal{B}$-coordinates for $x$ means solving the vector equation

$$x = c_1v_1 + c_2v_2 + \cdots + c_mv_m$$

in the unknowns $c_1, c_2, \ldots, c_m$. This (usually) means row reducing the augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_m & x \\ | & | & & | & | \end{array}\right).$$

Question: What happens if you try to find the $\mathcal{B}$-coordinates of *$x$ not in $V$*? You end up with an *inconsistent system*: $x = c_1v_1 + c_2v_2 + \cdots + c_mv_m$ has no solution.

# Extra: Why coefficients are unique

Lemma ← like a theorem, but less important

If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ **is a basis** for a subspace $V$, then any vector $x$ in $V$ can be written as a linear combination

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

for **unique coefficients** $c_1, c_2, \ldots, c_m$.

**Proof.** We know $x$ is a linear combination of the $v_i$ (they span $V$). *Suppose that* we *can write $x$* as a linear combination *with different lists of coefficients*:

$$\begin{aligned} x &= c_1 v_1 + c_2 v_2 + \cdots + c_m v_m \\ x &= c_1' v_1 + c_2' v_2 + \cdots + c_m' v_m \end{aligned}$$

Subtracting:

$$0 = x - x = (c_1 - c_1')v_1 + (c_2 - c_2')v_2 + \cdots + (c_m - c_m')v_m$$

Since $v_1, v_2, \ldots, v_m$ *are linearly independent*, they only have the trivial linear dependence relation. That *means each* $c_i - c_i' = 0$, or $c_i = c_i'$.