Section 3.2

Properties of Determinants

Plan for Today

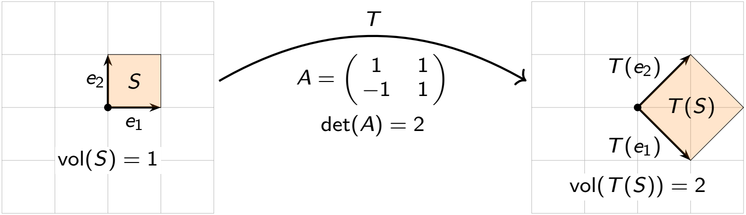
Last time, we gave a recursive formula for determinants in terms of cofactor expansions.

Plan for today:

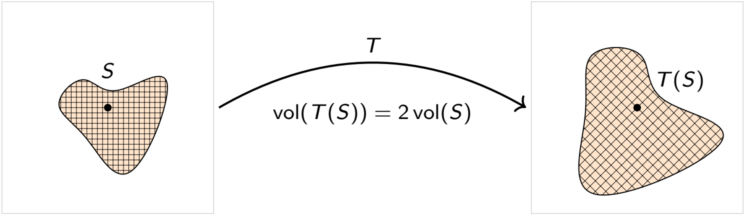
- An **abstract definition** of the determinant in terms of its properties.
- Computing determinants **using row operations**.
- Determinants and *products*: $\det(AB) = \det(A)\det(B)$,
- interpretation as volume,
- and linear transformations.

Linear Transformations and volumen

If S is the *unit cube*, then $T(S)$ is the parallelepiped formed by the columns of A. The **volumen changes** according to $\det(A)$.



For curvy regions: break S up into *tiny cubes*; each one is scaled by $|\det(A)|$. Then use *calculus* to reduce to the previous situation!



The Determinant is a Function

We can think of the *determinant as a function* of the entries of a matrix:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ \quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \end{array}$$

The formula for the determinant of an $n \times n$ matrix has $n!$ terms.

When mathematicians encounter a function whose *formula is too difficult* to write down, we try to **characterize it in terms of its properties**.

P characterizes object X

Not only does object X have property P,
but **X is the only one** thing that has property P.

Other example:

- e^x is unique function that has $f'(x) = f(x)$ and $f(0) = 1$.

Defining the Determinant in Terms of its Properties

Definition

The **determinant** is a function

$$\det\colon \{\text{square matrices}\} \longrightarrow \mathbf{R}$$

with the following **defining properties**:

1. $\det(I_n) = 1$
2. If we do a *row replacement* on a matrix, the determinant does not change.
3. If we *swap* two rows of a matrix, the determinant scales by -1.
4. If we *scale a row* of a matrix by k, the determinant scales by k.

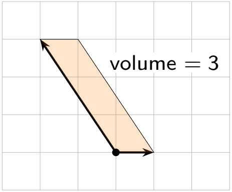
Why would we think of these properties? This is how volumes work!

1. The volume of the *unit cube* is 1.
2. Volumes don't change under *a shear*.
3. Volume of a *mirror image* is negative of the volume?
4. If you *scale one coordinate* by k, the volume is multiplied by k.

Properties of the Determinant

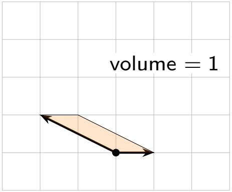
2×2 matrix

$$\det \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = 3$$



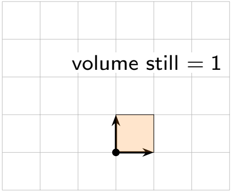
Scale: $R_2 = \frac{1}{3} R_2$

$$\det \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = 1$$



Row replacement: $R_1 = R_1 + 2R_2$

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$



Determinant for Elementary matrices

It is easy to calulate the determinant of an elementary matrix:

$$\det\begin{pmatrix}1 & 0 & 8\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \det(I_n) = 1 \qquad \text{(properties 1 and 2)}$$

$$\det\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix} = -\det(I_n) = -1 \qquad \text{(properties 1 and 3)}$$

$$\det\begin{pmatrix}1 & 0 & 0\\ 0 & 17 & 0\\ 0 & 0 & 1\end{pmatrix} = 17\det(I_n) = 17 \qquad \text{(properties 1 and 4)}$$

Poll

Poll

Suppose that A is a 4×4 matrix satisfying

$$Ae_1 = e_2 \quad Ae_2 = e_3 \quad Ae_3 = e_4 \quad Ae_4 = e_1.$$

What is $\det(A)$?

A. -1 B. 0 C. 1

These equations tell us the columns of A:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

You need 3 row swaps to transform this to the identity matrix.
So $\det(A) = (-1)^3 = -1$.

We can use the properties of the determinant and row reduction to compute the determinant of any matrix!

$$
\begin{aligned}
\det\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 5 & 7 & -4\end{pmatrix} &= -\det\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 5 & 7 & -4\end{pmatrix} && \text{(property 3)}\\
&= -\det\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 7 & -9\end{pmatrix} && \text{(property 2)}\\
&= -\det\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & -9\end{pmatrix} && \text{(property 2)}\\
&= (-1)\cdot(-9)\det\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} && \text{(property 4)}\\
&= (-1)\cdot(-9)\det\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} && \text{(property 2)}\\
&= 9 && \text{(property 1)}
\end{aligned}
$$

The second matrix is obtained from the first matrix by scaling by $-1/9$. So the determinant of the first matrix is -9 times the determinant of the second matrix.

Computing the Determinant by Row Reduction

We can use the properties of the determinant and row reduction to compute the determinant of any matrix!

Recall: The determinant of a *triangular matrix* is the product of the diagonal entries.

Saving some work We can stop row reducing when we get to row echelon form.

$$\det\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{pmatrix} = \cdots = -\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -9 \end{pmatrix} = 9.$$

Row reduction

This is almost always the easiest way to compute the determinant of a large, complicated matrix, either by hand or by computer.

Cofactor expansion is $O(n!) \sim O(n^n \sqrt{n})$, row reduction is $O(n^3)$.

Magical Properties of the Determinant

you really have to know these

1. det: {**square matrices**} → $\mathbf{R}$ is *the only function* satisfying the defining properties (1)–(4).
2. A is invertible if and only if $\det(A) \neq 0$.
3. If we row reduce A without row scaling, then
$$\det(A) = (-1)^{\#\text{swaps}}\big(\text{product of diagonal entries in REF}\big).$$
4. The determinant can be computed using any cofactor expansion.
5. $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = \det(A)^{-1}$.
6. $\det(A) = \det(A^T)$.
7. $|\det(A)|$ is the volume of the parallelepiped defined by the columns of A.
8. If A is an $n \times n$ matrix with transformation $T(x) = Ax$, and S is a subset of $\mathbf{R}^n$, then the volume of $T(S)$ is $|\det(A)|$ times the volume of S. (Even for curvy shapes S.)
9. The determinant is multi-linear (we'll talk about this next).

Multi-Linearity of the Determinant

Think of det as a function of the *columns* of an $n \times n$ matrix:

$$\det\colon \underbrace{\mathbf{R}^n \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n}_{n \text{ times}} \longrightarrow \mathbf{R}$$

$$\det(v_1, v_2, \ldots, v_n) = \det\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

Multi-linear: For *any* i and any vectors $v_1, v_2, \ldots, v_n$ and v_i' *and any scalar* c,

$$\det(v_1, \ldots, v_i + v_i', \ldots, v_n) = \det(v_1, \ldots, v_i, \ldots, v_n) + \det(v_1, \ldots, v_i', \ldots, v_n)$$
$$\det(v_1, \ldots, cv_i, \ldots, v_n) = c\det(v_1, \ldots, v_i, \ldots, v_n).$$

In words: if column i is a sum of two vectors v_i, v_i', then the determinant is the sum of two determinants, one with v_i in column i, and one with v_i' in column i.

Proof: just expand cofactors along column i.

- We already knew: Scaling *one column* by c scales det by c.
- *Same properties* hold if we replace column *by row*.
- *This only works one column (or row) at a time.*

Extra: Mathematical intricacies

The characterization of the determinant function in terms of its properties is very useful. It will *give us a fast way to compute* determinants and prove the other properties.

The **disadvantage** of defining a function by its properties *before having a formula*:

- how do you know such a *function exists*?
- *is there only one* function satisfying those properties?

In our case, we can compute the determinant of a matrix from its defining properties, so it exists. Why is it unique?

Extra: Intricacy applied

Why is Property 5 true? In Lay, there's a proof using elementary matrices. Here's another one.

Let B be an $n \times n$ matrix. There are two cases:

1. If $\det(B) = 0$, then B is not inverible. So for any matrix A, BA is not invertible. (Otherwise $B^{-1} = A(BA)^{-1}$.) So

$$\det(BA) = 0 = 0 \cdot \det(A) = \det(B)\det(A).$$

2. If A is invertible, define another function

$$f : \{n \times n \text{ matrices}\} \longrightarrow \mathbf{R} \quad \text{by} \quad f(B) = \frac{\det(BA)}{\det(A)}.$$

   Let's check the defining properties:

   1. $f(I_n) = \det(I_n A)/\det(A) = 1$.

   2–4. Doing a row operation on B and then multiplying by A, does the *same row operation* on BA. This is because a row operation is left-multiplication by an elementary matrix E, and $(EB)A = E(AB)$. Hence f scales like det with respect to row operations.

   By uniqueness, $f = \det$, i.e.,

$$\det(B) = f(B) = \frac{\det(AB)}{\det(A)} \quad \text{so} \quad \det(A)\det(B) = \det(AB).$$