Chapter 5

Eigenvalues and Eigenvectors

Section 5.1

Eigenvectors and Eigenvalues

Motivation: Difference equations

A Biology Question

How to predict a population of rabbits with given **dynamics**:

1. half of the newborn rabbits *survive* their first year;
2. of those, half *survive* their second year;
3. their maximum *life span* is three years;
4. Each rabbit gets $0, 6, 8$ *baby rabbits* in their three years, respectively.

Approach: Each year, count the population **by age**:

$$v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} \text{ where } \begin{cases} f_n & = \text{first-year rabbits in year } n \\ s_n & = \text{second-year rabbits in year } n \\ t_n & = \text{third-year rabbits in year } n \end{cases}$$

The *dynamics say*:

$$\overbrace{\begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}}^{v_{n+1}} = \begin{pmatrix} 6s_n + 8t_n \\ f_n/2 \\ s_n/2 \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}}^{Av_n}.$$

Motivation: Difference equations

Continued

This is a **difference equation**: $Av_n = v_{n+1}$

If you know *initial population* v_0, what happens *in 10 years* v_{10}?

Plug in a computer:

v_0	v_{10}	v_{11}
$\begin{pmatrix}1\\2\\3\end{pmatrix}$	$\begin{pmatrix}9459\\2434\\577\end{pmatrix}$	$\begin{pmatrix}19222\\4729\\1217\end{pmatrix}$
$\begin{pmatrix}3\\7\\9\end{pmatrix}$	$\begin{pmatrix}30189\\7761\\1844\end{pmatrix}$	$\begin{pmatrix}61316\\15095\\3881\end{pmatrix}$
$\begin{pmatrix}16\\4\\1\end{pmatrix}$	$\begin{pmatrix}16384\\4096\\1024\end{pmatrix}$	$\begin{pmatrix}32768\\8192\\2048\end{pmatrix}$

Notice any patterns?

1. Each segment of the population *essentially doubles* every year: $Av_{11} \approx 2v_{10}$.
2. The ratios get close to (16 : 4 : 1):

$$v_{11} \approx (\text{big}\#) \cdot \begin{pmatrix}16\\4\\1\end{pmatrix}.$$

New terms coming: eigenvalue, and eigenvector

Motivation: Difference equations

Continued (2)

We *want a formula* for vectors $v_0, v_1, v_2, \ldots$, such that

$$Av_0 = v_1 \qquad Av_1 = v_2 \qquad Av_2 = v_3 \qquad \ldots$$

We can see that $v_n = A^n v_0$. But multiplying by A each time is inefficient!

If v_0 satisfies $A_{v_0} = \lambda v_0$ then

$$v_n = A^{n-1}(Av_0) = \lambda A^{n-1} v_0 = \lambda^2 A^{n-2} v_0 \qquad \ldots = \lambda^n v_0.$$

It is **much easier** to compute $v_n = \lambda^{10} v_0$.

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad v_0 = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} \qquad Av_0 = 2v_0.$$

Starting with 16 baby rabbits, 4 first-year rabbits, and 1 second-year rabbit:

- The population will exactly double every year,
- In 10 years, you will have $2^{10} \cdot 16$ baby rabbits, $2^{10} \cdot 4$ first-year rabbits, and 2^{10} second-year rabbits.

Eigenvectors and Eigenvalues

Definition

Let A be an $n \times n$ matrix.

1. An **eigenvector** of A is a *nonzero vector* v in $\mathbf{R}^n$ such that $Av = \lambda v$, for some λ in $\mathbf{R}$. In other words, Av is a multiple of v.
2. We say that the *number* λ is the **eigenvalue for** v, and v is an **eigenvector for** λ.
3. Alternatively, λ in $\mathbf{R}$ is an eigenvalue of A if the equation $Av = \lambda v$ has a *nontrivial solution*.

Notes:

- Eigenvalues and eigenvectors are only for square matrices.
- Eigenvectors are by definition nonzero. *Eigenvalues may be equal to zero.*

Verifying Eigenvectors

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix} = 2v$$

Hence v is an *eigenvector* of A, with *eigenvalue* $\lambda = 2$.

Example

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4v$$

Hence v is an *eigenvector* of A, with *eigenvalue* $\lambda = 4$.

Poll

Poll

Which of the vectors

A. $\begin{pmatrix}1\\1\end{pmatrix}$ B. $\begin{pmatrix}1\\-1\end{pmatrix}$ C. $\begin{pmatrix}-1\\1\end{pmatrix}$ D. $\begin{pmatrix}2\\1\end{pmatrix}$ E. $\begin{pmatrix}0\\0\end{pmatrix}$

are eigenvectors of the matrix $\begin{pmatrix}1&1\\1&1\end{pmatrix}$?

$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=2\begin{pmatrix}1\\1\end{pmatrix}$	eigenvector with eigenvalue 2
$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=0\begin{pmatrix}1\\-1\end{pmatrix}$	eigenvector with eigenvalue 0
$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}=0\begin{pmatrix}-1\\1\end{pmatrix}$	eigenvector with eigenvalue 0
$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}3\\3\end{pmatrix}$	not an eigenvector
$\begin{pmatrix}0\\0\end{pmatrix}$	is never an eigenvector

Verifying Eigenvalues

Question: Is $\lambda = 3$ an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$?

In other words, does

$$\left.\begin{aligned} &Av = 3v \\ &Av - 3v = 0 \\ &(A - 3I)v = 0 \end{aligned}\right\} \text{have a nontrivial solution?}$$

We know how to answer that! Row reduction!

$$A - 3I = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

Parametric vector form: $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} -4 \\ 1 \end{pmatrix}$.

Then: Any nonzero multiple of $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ is an *eigenvector with eigenvalue $\lambda = 3$*

Check one of them:

$$\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix} = 3\begin{pmatrix} -4 \\ 1 \end{pmatrix}. \qquad ✓$$

Eigenspaces

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The **λ-eigenspace** of A is the set of all *eigenvectors of A with eigenvalue λ, plus the zero* vector:

$$\begin{aligned}\lambda\text{-eigenspace} &= \big\{v \text{ in } \mathbf{R}^n \mid Av = \lambda v\big\} \\ &= \big\{v \text{ in } \mathbf{R}^n \mid (A - \lambda I)v = 0\big\} \\ &= \operatorname{Nul}\big(A - \lambda I\big).\end{aligned}$$

The λ-eigenspace is a *subspace* of $\mathbf{R}^n$. How to find a basis? Parametric vector form!

Eigenspaces
Example

Find a basis for the $\underset{\lambda}{2}$-eigenspace of

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

$$A - 2I = \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{parametric vector form}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_2 \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

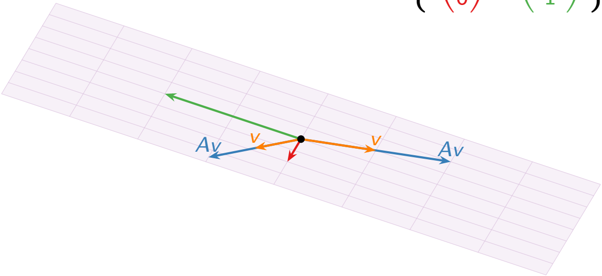
$$\xrightarrow{\text{basis}} \left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Eigenspaces

Picture

This is how eigenvalues and eigenvectors make matrices easier to understand.

What does this 2-eigenspace look like? A basis is $\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.



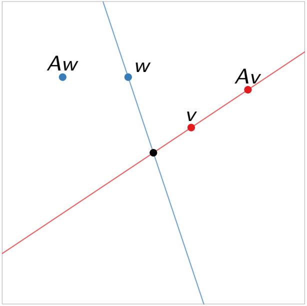
For any v in the 2-eigenspace, $Av = 2v$ by definition.
This means, on its 2-eigenspace, A acts by *scaling by 2*.

Geometrically

Eigenvectors

An eigenvector of a matrix A is *a nonzero vector v* such that:

- Av is a multiple of v, which means
- *Av is on the same line* as v.



v is an eigenvector

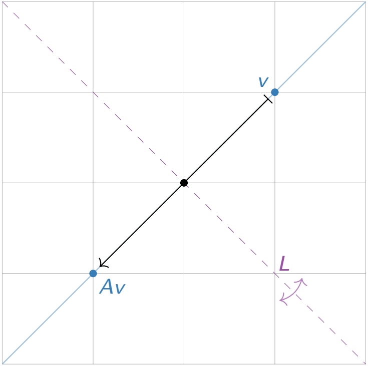
w is not an eigenvector

Eigenspaces

Geometry; example

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: Eigenvalues and eigenspaces of A? *No computations!*



Which vectors don't move off their line

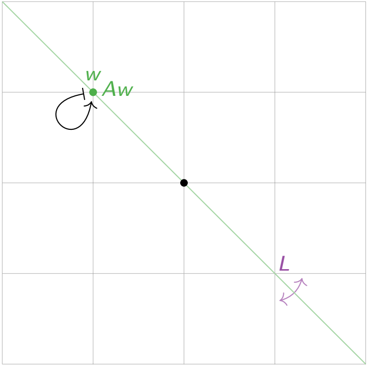
v is an eigenvector with eigenvalue -1.

Eigenspaces

Geometry; example

Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: Eigenvalues and eigenspaces of A? *No computations!*



Which vectors don't move off their line

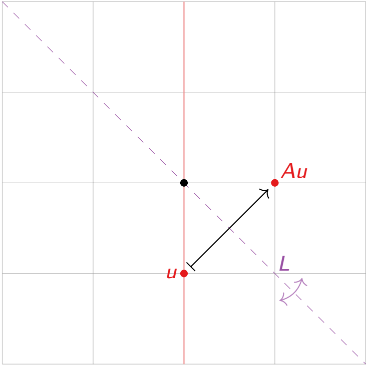
w is an eigenvector with eigenvalue 1.

Eigenspaces

Geometry; example

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: Eigenvalues and eigenspaces of A? *No computations!*



Which vectors don't move off their line

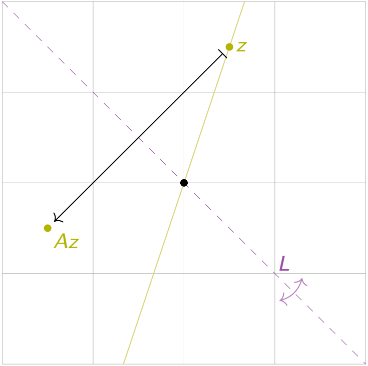
u is *not* an eigenvector.

Eigenspaces

Geometry; example

Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: Eigenvalues and eigenspaces of A? *No computations!*



Which vectors don't move off their line

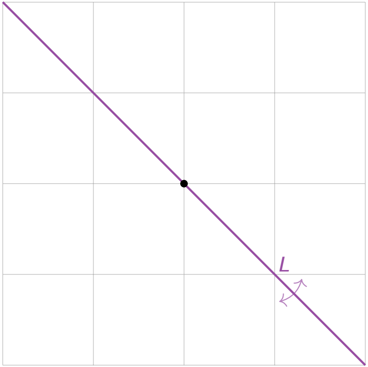
Neither is z.

Eigenspaces

Geometry; example

Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: Eigenvalues and eigenspaces of A? *No computations!*



Which vectors don't move off their line

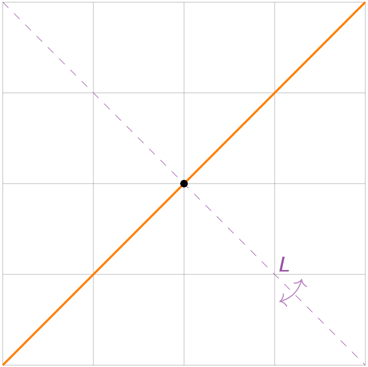
The 1-eigenspace is L
(all the vectors x where $Ax = x$).

Eigenspaces

Geometry; example

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: Eigenvalues and eigenspaces of A? *No computations!*



Which vectors don't move off their line

The (-1)-eigenspace is the line $y = x$
(all the vectors x where $Ax = -x$).

Eigenspaces

Summary

Let A be an $n \times n$ matrix and let λ be a number.

1. λ is an **eigenvalue of A**

   if and only if $(A - \lambda I)x = 0$ *has a nontrivial solution*,

   if and only if $\mathrm{Nul}(A - \lambda I) \neq \{0\}$.

2. Finding a basis for the **λ-eigenspace of A**

   means finding a basis for $\mathrm{Nul}(A - \lambda I)$ as usual, through

   the *general solution to* $(A - \lambda I)x = 0$ (parametric vector form).

3. The **eigenvectors** with eigenvalue λ are

   the nonzero elements of $\mathrm{Nul}(A - \lambda I)$

   that is, the *nontrivial solutions* to $(A - \lambda I)x = 0$.

Some facts you can work out yourself

Fact 1

A is **invertible** if and only if 0 *is not an eigenvalue* of A.

Fact 2

If $v_1, v_2, \dots, v_k$ are eigenvectors of A with **distinct eigenvalues** $\lambda_1, \dots, \lambda_k$, then $\{v_1, v_2, \dots, v_k\}$ is *linearly independent*.

Consequence of Fact 2

An $n \times n$ matrix has **at most** n distinct eigenvalues.

Why Fact 1?

$$\begin{aligned} 0 \text{ is an eigenvalue of } A &\iff Ax = 0 \text{ has a nontrivial solution} \\ &\iff A \text{ is not invertible.} \end{aligned}$$

Why Fact 2 (for two vectors)?

If v_2 is a multiple of v_1, then v_2 is contained in the λ_1-eigenspace. This is not true as v_2 does not have the same eigenvalue as v_1.

The Eigenvalues of a Triangular Matrix are the Diagonal Entries

- If we **know λ is eigenvalue**: easy to find eigenvectors (*row reduction*).
- And to **find all eigenvalues**? Will need to *compute a determinant*.

  Finding λ that has a non-trivial solution to $(A - \lambda I)v = 0$ boils down to finding λ that makes $\det(A - \lambda I) = 0$.

Theorem

The **eigenvalues** of a triangular matrix are the *diagonal entries*.

Example

Find all eigenvalues of $A = \begin{pmatrix} 3 & 4 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{pmatrix}$.

$$A - \lambda I = \begin{pmatrix} 3 & 4 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{pmatrix} - \lambda I_3 = \begin{pmatrix} 3-\lambda & 4 & 1 \\ 0 & -1-\lambda & -2 \\ 0 & 0 & 3-\lambda \end{pmatrix}$$

Since $\det(A - \lambda I) = (3 - \lambda)^2(-1 - \lambda)$, eigenvalues are $\lambda = 3$ and -1.