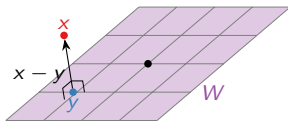


Section 6.2

Orthogonal Sets

Best Approximation

Due to measurement error, the measured x is not actually in the subspace it must lie on (*for theoretical reasons*).



Best approximation: y is the *closest point* to x on W .

Replace x with its orthogonal projection y onto W .

How do you know that y is the closest point?

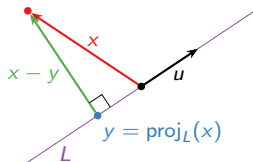
Orthogonal Projection onto a Line

Theorem

Let $L = \text{Span}\{u\}$ be a line in \mathbf{R}^n , and let x be in \mathbf{R}^n . The closest point to x on L is the point

$$\text{proj}_L(x) = \frac{x \cdot u}{u \cdot u} u.$$

This point is called the **orthogonal projection of x onto L** .



Choose term 'ortogonal' because $x - y$ is in L^\perp . That is, $u \cdot (x - y) = 0$:

$$u \cdot (x - y) = u \cdot \left(x - \frac{x \cdot u}{u \cdot u} u \right) = u \cdot x - \frac{x \cdot u}{u \cdot u} (u \cdot u) = u \cdot x - x \cdot u = 0.$$

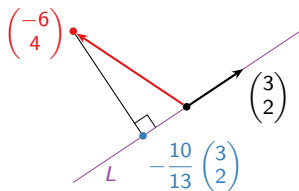
Orthogonal Projection onto a Line

Example

Compute the *orthogonal projection* of $x = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$ onto the line L spanned by

$$u = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

$$y = \text{proj}_L(x) = \frac{x \cdot u}{u \cdot u} u = \frac{-18 + 8}{9 + 4} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = -\frac{10}{13} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$



Orthogonal Sets

Definition

A set of *nonzero* vectors is **orthogonal** if each pair of vectors is orthogonal. Such set is **orthonormal** if, in addition, each vector is a *unit vector*.

Example: $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is an orthogonal set.

Check:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0 \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0 \quad \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0.$$

Orthogonal bases

Linearly independent

An *orthogonal set of vectors* $\mathcal{B} = \{u_1, u_2, \dots, u_m\}$ is linearly independent. Therefore **\mathcal{B} forms a basis** for $W = \text{Span } \mathcal{B}$.

Why?

Suppose $\{u_1, u_2, \dots, u_m\}$ is *orthogonal* and that

$$c_1 u_1 + c_2 u_2 + \dots + c_m u_m = 0$$

Now **dot-multiply by u_1** . We see that c_1 *must be zero*:

$$0 = u_1 \cdot (c_1 u_1 + c_2 u_2 + \dots + c_m u_m) = c_1(u_1 \cdot u_1) + 0 + 0 + \dots + 0.$$

Similarly for the other c_i 's (there is only trivial solution). Hence the set is linearly independent.

\mathcal{B} -coordinates for Orthogonal bases

Theorem

Let $\mathcal{B} = \{u_1, u_2, \dots, u_m\}$ be an orthogonal set, and let x be a vector in $W = \text{Span } \mathcal{B}$. Then

$$x = \sum_{i=1}^m \frac{x \cdot u_i}{u_i \cdot u_i} u_i = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m} u_m.$$

An advantage

For orthogonal bases, it's *easy to compute the \mathcal{B} -coordinates* of a vector x in W :

$$\left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_2 \cdot u_2}, \dots, \frac{x \cdot u_m}{u_m \cdot u_m} \right).$$

Why? If $x = c_1 u_1 + c_2 u_2 + \cdots + c_m u_m$, then

$$x \cdot u_1 = c_1(u_1 \cdot u_1) + 0 + \cdots + 0 \implies c_1 = \frac{x \cdot u_1}{u_1 \cdot u_1}.$$

Similarly for the other c_i 's.

Orthogonal Bases

Geometric reason

Theorem

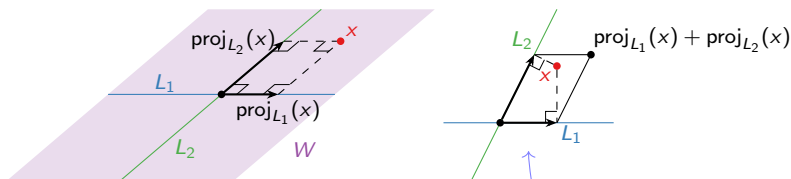
Let $\mathcal{B} = \{u_1, u_2, \dots, u_m\}$ be an orthogonal set, and let x be a vector in $W = \text{Span } \mathcal{B}$. Then

$$x = \sum_{i=1}^m \frac{x \cdot u_i}{u_i \cdot u_i} u_i = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \boxed{\frac{x \cdot u_2}{u_2 \cdot u_2} u_2} + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m} u_m.$$

proj_{L₂}(u₂)

If L_i is the line spanned by u_i , then this says

$$x = \text{proj}_{L_1}(x) + \text{proj}_{L_2}(x) + \cdots + \text{proj}_{L_m}(x).$$



Warning: This only works for an *orthogonal* basis.

Orthogonal Bases

Example

Problem: Find the \mathcal{B} -coordinates of $x = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, where

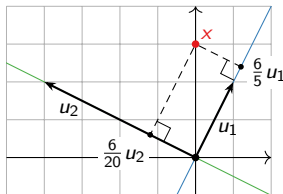
$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \end{pmatrix} \right\}.$$

Old way:

$$\left(\begin{array}{cc|c} 1 & -4 & 0 \\ 2 & 2 & 3 \end{array} \right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 6/5 \\ 0 & 1 & 6/20 \end{array} \right) \implies [x]_{\mathcal{B}} = \begin{pmatrix} 6/5 \\ 6/20 \end{pmatrix}.$$

New way: Exploit that \mathcal{B} is an *orthogonal basis*.

$$x = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{3 \cdot 2}{1^2 + 2^2} u_1 + \frac{3 \cdot 2}{(-4)^2 + 2^2} u_2 = \frac{6}{5} u_1 + \frac{6}{20} u_2.$$



$$\implies [x]_{\mathcal{B}} = \begin{pmatrix} 6/5 \\ 6/20 \end{pmatrix}.$$

Orthogonal Bases

Example

Problem: Find the \mathcal{B} -coordinates of $x = (6, 1, -8)$ where

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

Answer: Check that \mathcal{B} is *orthogonal basis*, then

$$\begin{aligned} [x]_{\mathcal{B}} &= \left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_2 \cdot u_2}, \frac{x \cdot u_3}{u_3 \cdot u_3} \right) \\ &= \left(\frac{6 \cdot 1 + 1 \cdot 1 - 8 \cdot 1}{1^2 + 1^2 + 1^2}, \frac{6 \cdot 1 + 1 \cdot (-2) - 8 \cdot 1}{1^2 + (-2)^2 + 1^2}, \frac{6 \cdot 1 + 1 \cdot 0 + (-8) \cdot (-1)}{1^2 + 0^2 + (-1)^2} \right) \\ &= \left(-\frac{1}{3}, -\frac{2}{3}, 7 \right). \end{aligned}$$

Check:

$$\begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix} = -\frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + 7 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}. \quad \checkmark$$