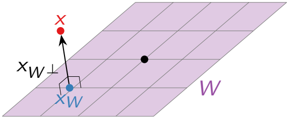
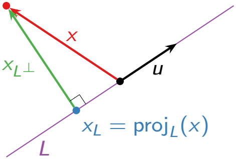
Section 6.3

Orthogonal Projections

Motivation



Example with a line: The closest point to x in L is $\text{proj}_L(x) = \frac{x \cdot u}{u \cdot u} u$



Let $u = \binom{3}{2}$ and let $L = \text{Span}\{u\}$. Let $x = \binom{-6}{4}$. In this case,

$$x_L = \text{proj}_L(x) = -\frac{10}{13}\begin{pmatrix}3\\2\end{pmatrix} \qquad x_{L^\perp} = x - \text{proj}_L(x) = \begin{pmatrix}-6\\4\end{pmatrix} + \frac{10}{13}\begin{pmatrix}3\\2\end{pmatrix}.$$

Orthogonal Projections

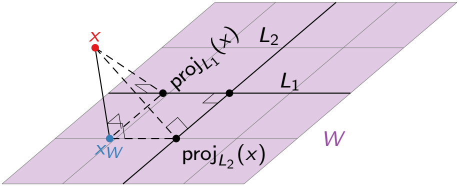
Definition

Let W be a subspace of $\mathbf{R}^n$, and let $\{u_1, u_2, \ldots, u_m\}$ be an *orthogonal* basis for W. The **orthogonal projection** of a vector x onto W is

$$\text{proj}_W(x) \overset{\text{def}}{=} \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i.$$

Note: If $L_i = \text{Span}\{u_i\}$. Then $\frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \text{proj}_{L_i}(x)$.

The orthogonal projection is formed by adding orthogonal projections onto perpendicular lines.



We can think of orthogonal projection as a *transformation*:

$$\text{proj}_W : \mathbf{R}^n \longrightarrow \mathbf{R}^n \qquad x \mapsto \text{proj}_W(x).$$

Theorem

Let W be a subspace of $\mathbf{R}^n$.

1. proj_W is a *linear* transformation.
2. For every x in W, we have $\text{proj}_W(x) = x$.
3. For every x in $W^\perp$, we have $\text{proj}_W(x) = 0$.
4. The range of proj_W is W.

The following is *the property we wanted* all along.

Best Approximation Theorem

Let W be a subspace of $\mathbf{R}^n$, and let x be a vector in $\mathbf{R}^n$. Then $y = \text{proj}_W(x)$ is *the closest point in W to x*, in the sense that

$$\text{dist}(x, y) \leq \text{dist}(x, y') \quad \text{for all} \quad y' \text{ in } W.$$

Best approximation

Every vector x can be *decompsed uniquely* as $x = x_W + x_{W^\perp}$ where

- $x_W = y$ is the *closest vector* to x *in* W, and
- $x_{W^\perp} = x - y$ is in $W^\perp$.

Theorem

Let W be a subspace of $\mathbf{R}^n$, and let x be a vector in $\mathbf{R}^n$. Then $\text{proj}_W(x)$ is the closest point to x in W. Therefore

$$x_W = \text{proj}_W(x) \qquad x_{W^\perp} = x - \text{proj}_W(x).$$

Why? Let $y = \text{proj}_W(x)$. We need to show that $x - y$ is in $W^\perp$. In other words, $u_i \cdot (x - y) = 0$ for each i. Let's do u_1:

$$u_1 \cdot (x - y) = u_1 \cdot \left(x - \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i} u_i \right) = u_1 \cdot x - \frac{x \cdot u_1}{u_1 \cdot u_1}(u_1 \cdot u_1) - 0 - \cdots = 0.$$

What is the matrix for $\text{proj}_W \colon \mathbf{R}^3 \to \mathbf{R}^3$, where

$$W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}?$$

Answer: Recall how to compute the matrix for a linear transformation:

$$A = \begin{pmatrix} | & | & | \\ \text{proj}_W(e_1) & \text{proj}_W(e_2) & \text{proj}_W(e_3) \\ | & | & | \end{pmatrix}.$$

We compute:

$$\text{proj}_W(e_1) = \frac{e_1 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{e_1 \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/6 \\ 1/3 \\ -1/6 \end{pmatrix}$$

$$\text{proj}_W(e_2) = \frac{e_2 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{e_2 \cdot u_2}{u_2 \cdot u_2} u_2 = 0 + \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$\text{proj}_W(e_3) = \frac{e_3 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{e_3 \cdot u_2}{u_2 \cdot u_2} u_2 = -\frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/6 \\ 1/3 \\ 5/6 \end{pmatrix}$$

Therefore $A = \begin{pmatrix} 5/6 & 1/3 & -1/6 \\ 1/3 & 1/3 & 1/3 \\ -1/6 & 1/3 & 5/6 \end{pmatrix}$.

Orthogonal Projections

Matrix facts

Let A **be the matrix for** proj_W, where W is an m-dimensional subspace of $\mathbf{R}^n$.

Facts:

1. A *is diagonalizable* with eigenvalues 0 and 1;
2. it is similar to the *diagonal matrix with m ones and $n - m$ zeros* on the diagonal, and
3. $A^2 = A$.

Example: If W is a plane in $\mathbf{R}^3$, then A is similar to projection onto the xy-plane:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Why 1-2? Let $v_1, v_2, \ldots, v_m$ be a basis for W, and let $v_{m+1}, v_{m+2}, \ldots, v_n$ be a basis for $W^\perp$. These are (linearly independent) eigenvectors with eigenvalues 1 and 0, respectively, and they form a basis for $\mathbf{R}^n$ because there are n of them.

Why 3? Projecting twice is the same as projecting once:

$$\text{proj}_W \circ \text{proj}_W = \text{proj}_W \implies A \cdot A = A.$$

Orthogonal Projections

Minimum distance

What is the (minimum) *distance from e_1 to W* $=$ Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$?

Answer: From e_1 to its closest point on W:

$$\text{dist}(e_1, \text{proj}_W(e_1)) = \|(e_1)_{W^\perp}\|.$$

$$\begin{aligned} &\text{dist}(e_1, \text{proj}_W(e_1)) \\ &= \left\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 5/6 \\ 1/3 \\ -1/6 \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} 1/6 \\ -1/3 \\ -1/6 \end{pmatrix} \right\| \\ &= \sqrt{(1/6)^2 + (-1/3)^2 + (-1/6)^2} \\ &= \frac{1}{\sqrt{6}}. \end{aligned}$$