Announcements

Tuesday, April 17

- Please fill out the CIOS form online.
  - It is important for me to get responses from most of the class: I use these for preparing future iterations of this course.
  - If we get an 80% response rate before the final, I'll drop the *two* lowest quiz grades instead of one.

- Office hours: Tuesdays-Thursdays 4–5pm, Clough 248
  - As always, TAs' office hours are posted on the website.
  - Math Lab at Clough is also a good place to visit.

- Grades:
  - All quiz and Midterm's grades are uploaded
  - Participation points will be posted during exam period
  - Optional Assignment: due by email on April 20th (midnight)

Section 7.3

Constrained Optimization

Motivation: How to allocate resources

Problem: The government wants to repair

- w_1 hundred miles of public roads
- w_2 hundred acres of parks

Resources are limited, so cannot work on more than

- 3 miles of roads or
- 2 acres of park;
- *general condition* is:

$$4w_1^2 + 9w_2^2 \leq 36$$

How to allocate resources?
Utility function: Considering overall benefits, want to maximize

$$q(w_1, w_2) = w_1 w_2.$$

(i.e.Do not focus solely on roads nor parks)

How would you *maximize utility* $q(w_1, w_2)$?

Constrained Optimization

Optimization problems

Maximize (or minimize) the value of a given function.

- This a broad and important area,
- here we only focus on *quadratic functions*.

Example

What is the maximum value possible for

- $Q(x) = 3x_1^2 + 3x_2^2$?
  under the constraint $||x||^2 = 1$
- $Q(x) = 3x_1^2 + 7x_2^2$?
  under the constraint $||x||^2 = 1$

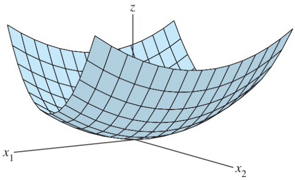


FIGURE 1 $z = 3x_1^2 + 7x_2^2$.

The constraint in these optimization problems

We will keep the restriction that vectors x in $\mathbf{R}^n$ have unit length;

$$||x|| = 1, \qquad x \cdot x = 1 \qquad x^T x = 1$$

or more commonly used: $x_1^2 + x_2^2 + \cdots x_n^2$.

Example

$Q(x) = 3x_1^2 + 7x_2^2$

Plot this function in 3-dimension as:

$$\begin{pmatrix} x_1 \\ x_2 \\ Q(x) \end{pmatrix}$$

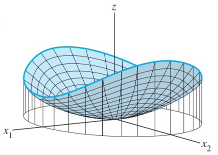


FIGURE 2 The intersection of $z = 3x_1^2 + 7x_2^2$ and the cylinder $x_1^2 + x_2^2 = 1$.

The constrained optimization problem

Given a **quadratic form $Q(x)$**, restricted to *unit vectors*,

What is the maximum and minimum values of $Q(x)$, which vectors attain such extremes?

An easy case: no cross-product

Example

If $Q(x) = 3x_1^2 + 7x_2^2$, and constraint is $x^T x = 1$.

- What are the *maximum and minimum* values of $Q(x)$,
- which *vectors* attain such values?

The constraint means $x_1^2 + x_2^2 = 1$, so for any such vector x:

- $Q(x) = 3x_1^2 + 7x_2^2 \leq 7x_1^2 + 7x_2^2 \leq 7(x_1^2 + x_2^2) = 7$

  $$\text{attained by vectors } \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- $Q(x) = 3x_1^2 + 7x_2^2 \geq 3x_1^2 + 3x_2^2 \geq 3(x_1^2 + x_2^2) = 3$

  $$\text{attained by vectors } \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Another easy case: no cross-product

Example

If $Q(x) = 2x_1^2 + 4x_2^2 + x^3$, and constraint is $x^T x = 1$.

- What are the *maximum and minimum* values of $Q(x)$,
- which *vectors* attain such values?

The associated matrix is $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with eigenvalues $1, 2, 4$.

- The maximum value is 4, attained by $\pm \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
- The minimum value is 1, attained by $\pm \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Eigenvalues

Answer seems: The largest and smallest eigenvalues (and eigenvectors) of A. How does it work in general?

The Constrained Optimization theorem

Theorem

Let A be a symmetric matrix and $Q(x) = x^T Ax$ a quadratic function

- **Maximum:** the maximum value of $Q(x)$ subject to $x^T x = 1$ equals the **largest eigenvalue M** of A.

  This maximum is attained by an eigenvector of A corresponding to M.
- *Minimum:* the minimum value of $Q(x)$ subject to $x^T x = 1$ equals the *smallest eigenvalue m* of A.

  This minimum is attained by an eigenvector of A corresponding to m.

How to use this information? To find maximum/minimum values of $Q(x)$, under *restriction $x^T x = 1$*:

- Find the eigenvalues of A, list them in decreasing order $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$.
- Then maximum is $M = \lambda_1$ and minimum is $m = \lambda_n$.

Why it works for all quadratic functions?

From Section 7.2

Recall all quadratic functions $Q(x)$ have

- A symmetric matrix associated A,
- An orthogonal diagonalization for $A = PDP^T$,
- *Form is equivalent to* $\widehat{Q}(y) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots \lambda_n y_n^2$,

  under a suitable change of variables $x = Py$

The problem for $\widehat{Q}(y)$: Maximum is largest of λ_i's, say λ_1. Then vector attaining maximum is e_1.

The problem for $Q(x)$: can use $\widehat{Q}(y)$ because P orthonormal!

The maximum is $\widehat{Q}(e_1) = Q(Pe_1) = \lambda_1$ attained by Pe_1 (first column of P).

Example

Example

What is the maximum value of $Q(x) = x^T Ax$ subject to $x^T x = 1$,
$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$.

For maximum value: compute the characteristic equation of A

$$\det(A - \lambda I) = 0 = (\lambda - 6)(\lambda - 3)(\lambda - 1).$$

Then the maximum value is 6.

For unit vector attaining $Q(x) = 6$: Find eigenvector of A corresponding to 6,

and normalize it!

Get both using a decompostion of A...

Have access to orthogonal diagonalization?

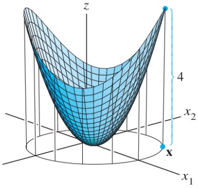
Example

What is the vector attaining the maximum value of $Q(x) = x^T Ax$ subject to $x^T x = 1$, $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

If you have the *orthogonal diagonalization of A*:

$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

- The **maximum** value is 4
- and the *vectors attaining* such value are $\pm \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$



The maximum value of $Q(\mathbf{x})$ subject to $\mathbf{x}^T\mathbf{x} = 1$ is 4.

Poll

Poll

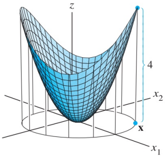
What is the vector attaining the maximum value of $Q(x) = x^T A x$ subject to $x^T x = 1$ and $x^T u = 0$, when

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \qquad u = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

$$A = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Using *orthogonal diagonalization of A*:

- With the new constraint the **maximum** value is 2
- and the *vectors attaining* such value are $\pm \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$



The maximum value of $Q(\mathbf{x})$ subject to $\mathbf{x}^T\mathbf{x} = 1$ is 4.

Additional constraints

Other eigenvalues/eigenvectors

Let A be a symmetric matrix. Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the eigenvalues of A listed in decreasing order.

Let $A = PDP^T$ be an orthogonal diagonalization where diagonal entries in D are $\lambda_1, \lambda_2, \ldots, \lambda_n$ and columns in P are $u_1, u_2, \ldots, u_n$.

- The maximum value of $x^T Ax$ subject to the constraints

$$x^T x = 1 \qquad \text{and} \qquad x^T u_1 = 0$$

  is the *second largest* eigenvalue λ_2, attained by $\pm u_2$.
- The maximum value of $x^T Ax$ subject to the constraints

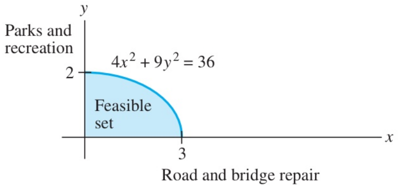
$$x^T x = 1 \qquad \text{and} \qquad x^T u_1 = 0, \qquad x^T u_2 = 0$$

  is the *third largest* eigenvalue λ_3, attained by $\pm u_3$.
- Can you see a pattern?

Back to the application: setup problem

Problem: The government wants to repair w_1 hundred miles of public roads and w_2 hundred acres of parks.

1. *Maximize* work done! constrain to $4w_1 + 9w_2 = 36$



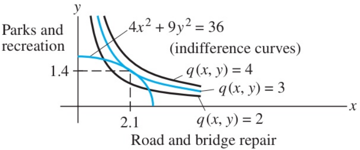


FIGURE 4 The optimum public works schedule is (2.1, 1.4).

2. Fit to quadratic optimization template (additional change of variables)

$$x_1 = \frac{w_1}{3} \qquad x_2 = \frac{w_2}{2}, \qquad \text{new constraint: } x_1^2 + x_2^2 = 1.$$

3. New utility function: subject to $x_1^2 + x_2^2 = 1$, maximize

$$Q(x_1, x_2) = (3x_1)(2x_2) = 6x_1x_2.$$

Back to the application: interpretation

The associated matrix A to $Q(x_1, x_2) = 6x_1x_2$ has orthogonal diagonalization:

$$A = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} = PDP^T;$$

with

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}.$$

This means:

- $Q(x_1, x_2)$ is **maximized when** $x_1 = x_2 = \frac{1}{\sqrt{2}}$,
- and the utility *function has value* 3.

Translates to:

- The utility function $q(w_1, w_2)$ is **maximized when** $w_1 = \frac{3}{\sqrt{2}}$ **and** $w_2 = \frac{2}{\sqrt{2}}$,
- and the utility function *has the same value 3*.