# MATH 2802
# FINAL PRACTICE EXAM

| Name | | Section | |
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Please **read all instructions** carefully before beginning.

- There are 10 problems in the exam and the maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- You may use the last page as scratch paper
- Good luck!

**1.** [2 points each] Circle **T** if the statement is always true and circle **F** if it is ever false. The matrices here are $n \times n$.

a) **T** **F** For a transformation $T : \mathbf{R}^4 \to \mathbf{R}^2$, $T$ is never one-to-one.

b) **T** **F** If $b$ is in the span of the columns of $A$, then $Ax = b$ is consistent.

c) **T** **F** If the columns of an $m \times n$ matrix $A$ form a linearly independent set then $Ax = b$ is consistent for any $b$ in $\mathbf{R}^m$.

d) **T** **F** If the geometric multiplicities of eigenvalues in $A$ sum up to $n$, then $A$ is diagonalizable.

e) **T** **F** A vector $v = (v_1, \dots, v_n)$ is steady-state vector of a stochastic matrix $A$ if $Av = v$ and the length $|v| = \sqrt{v_1^2 + \ldots + v_n^2} = 1$.

f) **T** **F** If $P$ is invertible, then $\det(PDP^{-1}) = \det(D)$.

g) **T** **F** Then dimension of the column space of $A$ is called $rank(A)$.

h) **T** **F** $A$ and $B = 2A$ are $n \times n$ matrices. If $\det(A) = 4$, then $\det(B) = 8$.

i) **T** **F** If $\{v_1, v_2, v_3\}$ is an orthogonal basis of $W$ then $\{v_1, v_2, 5v_3\}$ is an orthogonal basis for $W$.

j) **T** **F** The quadratic form $Q$ is positive definite if for all non-zero vectors $Q(x) \geq 0$.

**2.** Find the matrices corresponding to the following transformations $T, U : \mathbf{R}^2 \to \mathbf{R}^2$.

**a)** $T$ rotates by an angle of 180 degrees.

**b)** $U$ reflects through the $x = y$ line.

**c)** Evaluate $T \circ U(e_1)$ and $T \circ U(e_2)$

**3.** If $A = LU$ with $L$ and $U$ as below; solve the system of equations. $Ax = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(You must use the $LU$ factorization to receive full credit)

**4.** A car rental in Atlanta has a fleet of 500 cars and two locations, one at the airport and one at Midtown. A car rented at one location may be returned to either of the two locations. The administration has the following information: only 30% of the cars rented at the airport location are returned to the one at Midtown; while 50% of the cars rented at the Midtown location are returned to the airport location.

On Monday there were 300 cars at the airport and 200 cars were at the Midtown location. Let $x_9 = \begin{pmatrix} 300 \\ 200 \end{pmatrix}$ denote the distribution of cars from the rental on February 9th, $x_{10}$ to the distribution of cars from the rental on February 10th, $x_{11}$ to the distribution of cars from the rental on February 11th, and so on.

**a)** Provide the matrix $A$ that estimates $x_{10} = Ax_9$.

**b)** Provide the matrix equation that estimates $x_{12}$ as a product of a power of $A$ and $x_9$. *(Don't need to justify why)*

**c)** If $x_{28}$ is the distribution of cars from the rental on February 28th, write a matrix equation that estimates $x_{28}$ as a product of a power of $A$ and $x_9$. *(Don't need to justify why)*

**5.** In this problem, show your work and justify your answers.

**a)** Is 5 an eigenvalue of $A = \begin{pmatrix} -2 & 7 \\ -5 & 10 \end{pmatrix}$?

**b)** Is $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ an eigenvector of $B = \begin{pmatrix} 7 & 6 & -1 \\ 0 & 4 & 8 \\ 3 & -8 & 17 \end{pmatrix}$?

**c)** Find the 3-eigenspace of $C = \begin{pmatrix} 4 & -7 & 0 & 2 \\ 0 & 3 & -4 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

**d)** Is $C$ diagonalizable?

**6.** Consider a matrix $A$ which is row-equivalent to:

$$A = \begin{pmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 8 & 0 & 5 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**a)** Find a basis for $Col(A)$

**b)** What is the dimension for $Nul(A)$?

**c)** Justify your answer in b) *(There are several possible correct answers)*

**d)** If the dimension of $V$ is $m$ then:

- Any $m$ linearly independent vectors in V form ____________ for V .
- Any $m$ vectors that ____________ form ____________ for V .

**7.** There is a car rental with locations at the airport, Midtown and Marietta. After a comprehensive study the administration knows that:

- A car rented at the airport has 20% chance of being returned to the Midtown location and 10% chance of being returned to the Marietta location.
- A car rented at the Midtown location has 10% chance of being returned to the airport and 10% chance of being returned to the Marietta location.
- A car rented at the Marietta location has 30% chance of being returned to the airport and 30% chance of being returned to the Midtown location.

**a)** Find the transition matrix $Q$ (make transitions from airport, midtown and Marietta correspond to columns 1,2 and 3 respectively).

**b)** If the steady-state vector of $Q$ is $w = \frac{1}{28}\begin{pmatrix} 9 \\ 15 \\ 4 \end{pmatrix}$. What percentage of time will a car in the rental be returned to the Midtown location?

**8.** Let $W = Span\{v_1, v_2\}$ with $v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Find $y = proj_W \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$. Justify your answer.

**9.** Find the best approximation to $\begin{pmatrix}1\\1\\1\end{pmatrix}$ by vectors of the form $c_1\begin{pmatrix}1\\0\\0\end{pmatrix}+c_2\begin{pmatrix}0\\1\\0\end{pmatrix}$. What is the error in your approximation?

**10.** Let $Q(x_1,x_2)=3x_1^2+2x_1x_2+3x_2^2$. There is a matrix $P$ such that the change of variables $y=P^{-1}x$ gives the following form $Q(y)=ay_1^2+by_2^2$.

**a)** Find the matrix associated to $Q(x)=x^TAx$

**b)** Find an orthogonal decomposition for $A=PDP^T$

**c)** Give the values of $a$ and $b$

**d)** Is the quadratic form $Q(x)$ indefinite? Justify your answer.