Review for Chapter 1

Selected Topics

Linear Equations

We have four *equivalent ways of writing linear systems*:

1. As a system of equations:

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5 \end{aligned}$$

2. As an augmented matrix:

$$\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right)$$

3. As a vector equation ($x_1 v_1 + \cdots + x_n v_n = b$):

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a matrix equation ($Ax = b$):

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

In particular, *all four have the same solution set*.

Number of Solutions

There are *three possibilities* for the reduced row echelon form of the augmented matrix of a linear system.

1. The last column is a pivot column.
   There are *zero* solutions, i.e. the solution set is *empty*. In this case, the system is called **inconsistent**. Picture:

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

2. Every column except the last column is a pivot column.
   In this case, the system has a *unique solution*. Picture:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array}\right)$$

3. The last column is not a pivot column, and some other column isn't either.
   In this case, the system has *infinitely many solutions*, corresponding to the infinitely many possible values of the free variable(s). Picture:

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

Parametric Form of Solution Sets

To find the solution set to $Ax = b$, first form the augmented matrix $(\, A \mid b \,)$, then row reduce.

$$\left(\begin{array}{ccccc|c} 1 & 3 & 0 & 4 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -7 \end{array}\right)$$

This translates into

$$\begin{aligned} x_1 + 3x_2 \qquad\quad + x_4 \qquad &= 2 \\ x_3 - x_4 \qquad &= 3 \\ x_5 &= -7 \end{aligned}$$

The variables correspond to the non-augmented columns of the matrix.
The *free variables* correspond to the *non-augmented columns without pivots*.

Move the free variables to the other side, get the **parametric form**:

$$\begin{aligned} x_1 &= 2 - 3x_2 - x_4 \\ x_3 &= 3 \qquad\quad + x_4 \\ x_5 &= -7 \end{aligned}$$

This is a **solution for every value of** x_3 and x_4.

Span

The **span** of vectors $v_1, v_2, \ldots, v_n$ is the *set of all linear combinations of these vectors*:

$$\text{Span}\{v_1, v_2, \ldots, v_n\} = \left\{a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \mid a_1, a_2, \ldots, a_n \text{ in } \mathbf{R}\right\}.$$

Theorem

Let $v_1, v_2, \ldots, v_n$, and b be vectors in $\mathbf{R}^m$, and let A be the $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$. The following *are equivalent*:

either they're all true, or they're all false, for the given vectors

1. $Ax = b$ **is consistent.**
2. $(\,A \mid b\,)$ does not have a pivot in the last column.
3. b is in $\text{Span}\{v_1, v_2, \ldots, v_n\}$ (the span of the columns of A).

In this case, a solution to the matrix equation

$$A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b \quad \text{gives the linear combination} \quad x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b.$$

Parametric Vector Form of Solution Sets

Parametric form:

$$\begin{aligned} x_1 &= 2 - 3x_2 - x_4 \\ x_3 &= 3 + x_4 \\ x_5 &= -7 \end{aligned} \quad \xrightarrow{\text{add free variables}} \quad \begin{aligned} x_1 &= 2 - 3x_2 - x_4 \\ x_2 &= x_2 \\ x_3 &= 3 + x_4 \\ x_4 &= x_4 \\ x_5 &= -7 \end{aligned}$$

Now collect all of the equations into a vector equation:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \\ -7 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

This is the **parametric vector form** of the solution set. This means that the

$$(\text{solution set}) = \begin{pmatrix} 2 \\ 0 \\ 3 \\ 0 \\ -7 \end{pmatrix} + \text{Span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Homogeneous and Non-Homogeneous Equations

The equation $Ax = b$ is called **homogeneous** if $b = 0$, and *non-homogeneous otherwise*. A homogeneous equation always has the **trivial solution** $x = 0$:

$$A0 = 0.$$

The solution set to a homogeneous equation is **always a span**:

$$(\text{solutions to } Ax = 0) = \text{Span}\{v_1, v_2, \ldots, v_r\}$$

where r is the number of free variables. The solution set to a *consistent non-homogeneous* equation is

$$(\text{solutions to } Ax = b) = p + \text{Span}\{v_1, v_2, \ldots, v_r\}$$

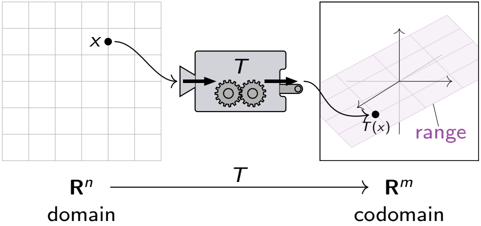
where *p is a specific solution* (i.e. some vector such that $Ap = b$), and $\text{Span}\{v_1, \ldots, v_r\}$ is the solution set to the homogeneous equation $Ax = 0$. This is a *translate of a span*.

Both expressions can be read off from the parametric vector form.

Definition

A **transformation** (or **function** or **map**) from $\mathbf{R}^n$ to $\mathbf{R}^m$ *is a rule* T that assigns to each vector *x in* $\mathbf{R}^n$ a vector *$T(x)$ in* $\mathbf{R}^m$.

Picture and vocabulary words:



It is **one-to-one** if *different vectors* in the domain go *to different vectors* in the codomain: $x \neq y \implies T(x) \neq T(y)$.

It is **onto** if *every vector* in the codomain *is $T(x)$ for some x*. In other words, the range equals the codomain.

Linear Transformations

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if it satisfies:

$$T(u+v) = T(u)+T(v) \quad \text{and} \quad T(cu) = cT(u)$$

for every u, v in $\mathbf{R}^n$ and every c in $\mathbf{R}$.

> Linear transformations are the same as matrix transformations.

Dictionary

Linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ $\rightsquigarrow$ $m \times n$ matrix $A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}$

$T(x) = Ax$

As always, $e_1, e_2, \ldots, e_n$ are the **unit coordinate vectors**

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Linear Transformations and Matrices

Let A be an $m \times n$ matrix and T be the **linear transformation** $T(x) = Ax$.

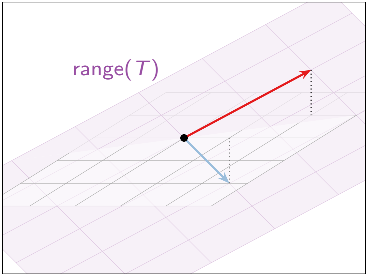
- The domain of T is $\mathbf{R}^n$: Input vector has n entries.
- The codomain of T is $\mathbf{R}^m$: Output vector has m entries.
- The *range of T is span of the columns of A*:
  This is the set of all b in $\mathbf{R}^m$ such that $Ax = b$ has a solution.

Example

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} \qquad T(x) = Ax$$

- The domain of T is $\mathbf{R}^2$.
- The codomain of T is $\mathbf{R}^3$.
- The range of T is

$$\text{Span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$



$\mathbf{R}^3 = \text{codomain}(T)$

Linear Independence

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is **linearly independent** if

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0 \quad \text{only when} \quad a_1 = a_2 = \cdots = a_n = 0.$$

Otherwise they are *linearly dependent*, and an equation $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$ with *some* $a_i \neq 0$ is a linear dependence relation.

Theorem

Let $v_1, v_2, \ldots, v_n$ be vectors in $\mathbf{R}^m$, and let A be the $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$. The following *are equivalent*:

1. The set $\{v_1, v_2, \ldots, v_n\}$ is **linearly independent**.
2. For every i between 1 and n, v_i is not in $\mathrm{Span}\{v_1, v_2, \ldots, v_{i-1}\}$.
3. $Ax = 0$ only has the trivial solution.
4. A has a pivot in every column.

If the *vectors are linearly dependent*, a *nontrivial solution* to the matrix equation

$$A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0 \quad \text{gives the linear dependence relation} \quad x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0.$$

Criteria on Linear Transformation: One-to-One

Theorem

Let A be an $m \times n$ matrix, and let $T : \mathbf{R}^n \to \mathbf{R}^m$ be the linear transformation $T(x) = Ax$. The following *are equivalent*:

1. **T is one-to-one**.
2. $T(x) = b$ has *one or zero solutions* for every b in $\mathbf{R}^m$.
3. $Ax = b$ has a *unique solution or is inconsistent* for every b in $\mathbf{R}^m$.
4. $Ax = 0$ has a unique solution.
5. The columns of A are *linearly independent*.
6. A has a *pivot in each column*.

Moral: If A has a pivot in each column then its reduced row echelon form looks like this:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and } (\, A \mid b \,) \text{ reduces to this: } \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \\ 0 & 0 & 0 & \star \end{array}\right).$$

This can be inconsistent, but if it is consistent, it has a unique solution.

Refer: slides for §1.4, §1.8, §1.9.

Criteria on Linear Transformation: Onto

Theorem

Let A be an $m \times n$ matrix, and let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be the linear transformation $T(x) = Ax$. The following *are equivalent*:

1. **T is onto**.
2. $T(x) = b$ has a *solution for every b* in $\mathbf{R}^m$.
3. $Ax = b$ is *consistent for every b* in $\mathbf{R}^m$.
4. The columns of A *span* $\mathbf{R}^m$.
5. A has a *pivot in each row*.

Moral: If A has a pivot in each row then its reduced row echelon form looks like this:

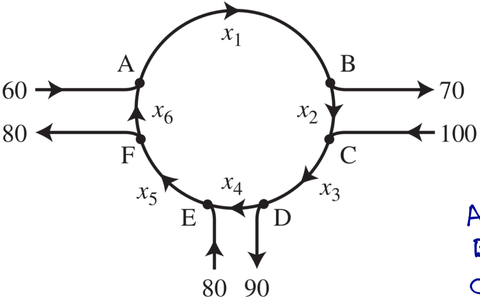
$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{pmatrix} \quad \text{and } (\, A \mid b \,) \text{ reduces to this: } \quad \left(\begin{array}{ccccc|c} 1 & 0 & \star & 0 & \star & \star \\ 0 & 1 & \star & 0 & \star & \star \\ 0 & 0 & 0 & 1 & \star & \star \end{array}\right).$$

There's no b that makes it inconsistent, so there's always a solution.

Refer: slides for §1.4 and §1.9.

Zero-sum systems: Traffic flows

- Zero-sum property of *flows in/out* around every node.



$$
\begin{aligned}
A&: \quad -x_1 + x_6 = -60\\
B&: \quad x_1 - x_2 = +70\\
C&: \quad x_2 - x_3 = -100\\
D&: \quad x_3 - x_4 = 90\\
E&: \quad x_4 - x_5 = -80\\
F&: \quad x_5 - x_6 = 80
\end{aligned}
$$

- **Constraints:** If the flow directions are mandatory (as traffic lanes) then their *values must remain non-negative*.

Linear combinations: Constructing a Weight-Loss Nutritious Diet

Or for any system of balanced/reliable supplies from different distributors.

Amounts (g) Supplied per 100 g of Ingredient				Amounts (g) Supplied by Cambridge Diet in One Day
Nutrient	Nonfat milk	Soy flour	Whey	
Protein	36	51	13	33
Carbohydrate	52	34	74	45
Fat	0	7	1.1	3

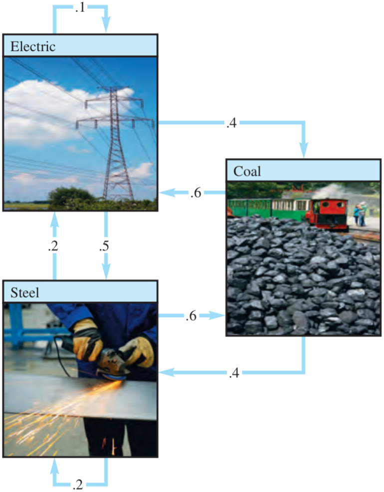
$$A = \begin{pmatrix} 36 & 51 & 13 \\ 52 & 34 & 74 \\ 0 & 7 & 1.1 \end{pmatrix} \qquad x = \begin{pmatrix} \text{ml. Nonfat milk} \\ \text{ml. Soy Flour} \\ \text{ml. Whey} \end{pmatrix}$$

Ax is the linear combination that represents the total amount of nutrients contained in the diet of x

- *Constraints:* The vector of weights for each product/distributor has to be non-negative to make sense in its context.

Linear combinations: Economy equilibrium prices

- **Equilibrium prices** can be assigned to the total outputs of the various sectors in such a way that the income of each sector balances its expenses
- *Constraints:* The prices have to be positive to make sense in this context.



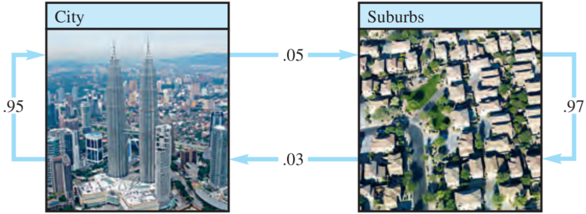
Distribution of Output from:			
Coal	**Electric**	**Steel**	**Purchased by:**
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

Equilibrium prices satisfy:

$$p_C = 0p_C + .4p_E + .6p_S$$
$$p_E = .6p_C + .1p_E + .2p_S$$
$$p_S = .4p_C + .5p_E + .2p_S$$

Difference equations: Population migration

This is a **difference equation**: $Ax_n = x_{n+1}$



Annual percentage migration between city and suburbs.

$$\begin{array}{cc} & \text{From:} \\ & \begin{array}{cc}\text{City} & \text{Suburbs}\end{array} \quad \text{To:} \\ & \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix} \quad \begin{array}{l}\text{City} \\ \text{Suburbs}\end{array} \end{array}$$

Pop. in 2017   city: $x_1 = 100$
suburbs: $x_2 = 80$

Pop in 2018: $A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

If you know *initial population* x_0, what happens *in 10 years* x_{10}?

Difference equations: Population growth

How to predict a population of rabbits with given **dynamics**:

1. half of the newborn rabbits *survive* their first year;
2. of those, half *survive* their second year;
3. their maximum *life span* is three years;
4. Each rabbit gets $0, 6, 8$ *baby rabbits* in their three years, respectively.

Approach: Each year, count the population **by age**:

$$v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} \quad \text{where} \quad \begin{cases} f_n & = \text{first-year rabbits in year } n \\ s_n & = \text{second-year rabbits in year } n \\ t_n & = \text{third-year rabbits in year } n \end{cases}$$

The *dynamics say*:

$$\overbrace{\begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}}^{v_{n+1}} = \begin{pmatrix} 6s_n + 8t_n \\ f_n/2 \\ s_n/2 \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}}^{Av_n}.$$