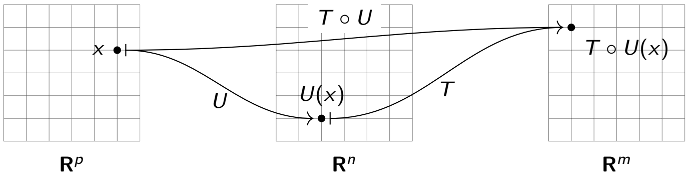
Review for Chapter 2

Selected Topics

Matrix Multiplication/Inversion and Linear Transformations

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ and $U\colon \mathbf{R}^p \to \mathbf{R}^n$ be linear transformations with matrices A and B. The **composition** is the linear transformation

$$T \circ U\colon \mathbf{R}^p \to \mathbf{R}^m \quad \text{defined by} \quad T \circ U(x) = T(U(x)).$$



Fact: The matrix for $T \circ U$ is AB.

Now let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be an *invertible* linear transformation. This means there is a linear transformation $T^{-1}\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $T \circ T^{-1}(x) = x$ for all x in $\mathbf{R}^n$. Equivalently, it means T is one-to-one and onto.

Fact: If A is the matrix for T, then A^{-1} is the matrix for T^{-1}.

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ scale the x-axis by 2, and let $U\colon \mathbf{R}^2 \to \mathbf{R}^2$ be counterclockwise rotation by 90°.

Their matrices are:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The composition $T \circ U$ is: first rotate counterclockwise by 90°, then scale the x-axis by 2. The matrix for $T \circ U$ is

$$AB = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}.$$

The inverse of U rotates *clockwise* by 90°. The matrix for U^{-1} is

$$B^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Matrix Inverses

The **inverse** of an $n \times n$ matrix A is a matrix A^{-1} such that $AA^{-1} = I_n$ (equivalently, $A^{-1}A = I_n$).

2×2 case:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Longrightarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

$n \times n$ case: Row reduce the augmented matrix $(\, A \mid I_n \,)$. If you get $(\, I_n \mid B \,)$, then $B = A^{-1}$. Otherwise, A is not invertible.

Solving linear systems by "dividing by A": If A is invertible, then

$$Ax = b \iff x = A^{-1}b.$$

Important

If A is invertible, then $Ax = b$ has exactly one solution for any b, namely, $x = A^{-1}b$.

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Example

Solve $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} x = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

Answer:

$$x = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{2 \cdot 3 - 1 \cdot 1} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3b_1 - b_2 \\ -b_1 + 2b_2 \end{pmatrix}$$

Elementary Matrices

Definition

An **elementary matrix** is a square matrix E which differs from I_n by one row operation.

There are three kinds:

scaling $(R_2 = 2R_2)$	row replacement $(R_2 = R_2 + 2R_1)$	swap $(R_1 \longleftrightarrow R_2)$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Fact: if E is the elementary matrix for a row operation, then EA differs from A by the same row operation.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \end{pmatrix} \rightsquigarrow B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 4 \end{pmatrix}$$

You get B by subtracting $2\times$ the first row of A from the second row.

$$B = EA \quad \text{where} \quad E = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad \begin{pmatrix} \text{subtract } 2\times \text{ the first row} \\ \text{of } I_2 \text{ from the second row} \end{pmatrix}.$$

The Inverse of an Elementary Matrix

Fact: the inverse of an elementary matrix E is the elementary matrix obtained by doing the opposite row operation to I_n.

$$\underset{R_2 = R_2 \times 2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}} = \underset{R_2 = R_2 \div 2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \qquad \underset{R_2 = R_2 + 2R_1}{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}} = \underset{R_2 = R_2 - 2R_1}{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\underset{R_1 \longleftrightarrow R_2}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}} = \underset{R_1 \longleftrightarrow R_2}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

If A is invertible, then there are a sequence of row operations taking A to I_n:

$$E_r E_{r-1} \cdots E_2 E_1 A = I_n.$$

Taking inverses (note the order!):

$$A = E_1^{-1} E_2^{-1} \cdots E_r^{-1} I_n = E_1^{-1} E_2^{-1} \cdots E_r^{-1}.$$

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix, and let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. A is row equivalent to I_n.
4. A has n pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of A are linearly independent.
7. T is one-to-one.
8. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
9. The columns of A span $\mathbf{R}^n$.
10. T is onto.
11. A has a left inverse (there exists B such that $BA = I_n$).
12. A has a right inverse (there exists B such that $AB = I_n$).
13. A^T is invertible.
14. The columns of A form a basis for $\mathbf{R}^n$.
15. $\operatorname{Col} A = \mathbf{R}^n$.
16. $\dim \operatorname{Col} A = n$.
17. $\operatorname{rank} A = n$.
18. $\operatorname{Nul} A = \{0\}$.
19. $\dim \operatorname{Nul} A = 0$.

Learn it!

LU factorization

A guru provides, for (suitable) $m \times n$ matrix A, matrices L and U such that

- L is lower triangular $m \times m$ matrix (with *ones on the diagonal*)
- U is an $m \times n$ row echelon form (not necessary reduced)
- $A = LU$

E.g.

$$A = \begin{pmatrix} 2 & 4 & -1 & 5 & -1 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

If $A = LU$ and want to solve $Ax = b$

1. Solve $Ly = b$,
2. Solve $Ux = y$,
3. Now, x **is a solution** to $Ax = b$.

Construction of LU factorizations

Our assumption: Row reduction of A requires *no row sawps*

1. *Row reduce* the matrix A, but *do not normalize pivots* to being one's; e.g.
2. Separate the row reduction according to 'clearing' pivot columns
3. From row reduction, gather 'vectors' below pivots and normalize. These form the 'vectors' in L.
4. If there are more columns in L than pivots, then leave rest of entries blank.

1.-2.

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} = A_1$$

$$\sim A_2 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U$$

3.

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 5 \end{bmatrix}$$

$$\div 2 \downarrow \quad \div 3 \downarrow \quad \div 2 \downarrow \quad \div 5 \downarrow$$

$$L = \begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

A second example

Find the LU factorization of A:

$$A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & -3 & -1 & 6 \\ 0 & 6 & 2 & -7 \\ 0 & -9 & -3 & 13 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$\longrightarrow \begin{bmatrix} 2 \\ 6 \\ 2 \\ 4 \\ -6 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 6 \\ -9 \end{bmatrix} \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix}$$

$$\div 2 \qquad \div 3 \qquad \div 5$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\begin{bmatrix} 1 & & & \\ 3 & 1 & & \\ 1 & -1 & 1 & \cdots \\ 2 & 2 & -1 & \\ -3 & -3 & 2 & \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 1 & 0 \\ -3 & -3 & 2 & 0 & 1 \end{bmatrix}$$