Review for Midterm 2

Selected Topics

Subspaces

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Examples:

- Any $\mathrm{Span}\{v_1, v_2, \ldots, v_m\}$.
- The *column space* of a matrix: $\mathrm{Col}\,A = \mathrm{Span}\{\text{columns of } A\}$.
- The *null space* of a matrix: $\mathrm{Nul}\,A = \{x \mid Ax = 0\}$.
- $\mathbf{R}^n$ and $\{0\}$

If V can be written in any of the above ways, then it is automatically a subspace: you're done!

Example

Is $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x + y = 0 \right\}$ a subspace?

1. Since $0 + 0 = 0$, the zero vector is in V.
2. Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ be arbitrary vectors in V.
   - This means $x + y = 0$ and $x' + y' = 0$.
   - We have to check if $\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \\ z + z' \end{pmatrix}$ is in V.
   - This means $(x + x') + (y + y') = 0$.

   Indeed:

   $$(x + x') + (y + y') = (x + y) + (x' + y') = 0 + 0 = 0,$$

   so condition (2) holds.

Example

Is $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x + y = 0 \right\}$ a subspace?

3. Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be in V and let c be a scalar.
   - This means $x + y = 0$.
   - We have to check if $c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix}$ is in V.
   - This means $cx + cy = 0$.

   Indeed:
   $$cx + cy = c(x + y) = c \cdot 0 = 0.$$
   So condition (3) holds.

Since conditions (1), (2), and (3) hold, V is a subspace.

Basis of a Subspace

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbf{R}^n$ such that:

1. $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

To check that $\mathcal{B}$ is a basis for V, you have to check two things:

1. $\mathcal{B}$ spans V.
2. $\mathcal{B}$ is linearly independent.

This is what it means to justify the statement "$\mathcal{B}$ is a basis for V."

Basis Theorem

Let V be a subspace of dimension m. Then:

- Any m linearly independent vectors in V form a basis for V.
- Any m vectors that span V form a basis for V.

So if you *already know the dimension* of V, you only have to check *one*.

Verify that $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x + y = 0 \right\}$.

0. In V: both are in V because $1 + (-1) = 0$ and $0 + 0 = 0$.
1. Span: If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in V, then $y = -x$, so we can write it as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -x \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

2. Linearly independent:

$$x \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \implies \begin{pmatrix} x \\ -x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies x = y = 0.$$

Knew a priori that $\dim V = 2$*:* then only have to check 0, then 1 *or* 2.

Bases of Col A and Nul A

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis $\leftsquigarrow$ pivot columns in rref

So a basis for Col A is $\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}$. A vector in Col A : $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

Parametric vector form for solutions to $Ax = 0$:

$$x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\substack{\text{basis of} \\ \text{Nul}\, A}} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

A vector in Nul A: any solution to $Ax = 0$, e.g., $x = \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}$.

Rank Theorem

Rank Theorem

If A is an $m \times n$ matrix, then

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n = \text{the number of columns of } A.$$

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

basis of Col A

free variables

In this case, $\operatorname{rank} A = 2$ and $\dim \operatorname{Nul} A = 2$, and $2 + 2 = 4$, which is the number of columns of A.

1. Special formulas for 2×2 and 3×3 matrices.
2. For [upper or lower] triangular matrices:

$$\det A = (\text{product of diagonal entries}).$$

3. Cofactor expansion along any row or column:

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \text{ for any fixed } i$$

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} \text{ for any fixed } j$$

   Start here for matrices with a row or column with lots of zeros.
4. By row reduction without scaling:

$$\det(A) = (-1)^{\#\text{swaps}} \big(\text{product of diagonal entries in REF}\big)$$

   This is fastest for big and complicated matrices.
5. Cofactor expansion and any other of the above. (The cofactor formula is recursive.)

Definition

The **determinant** is a function

$$\det\colon \{\text{square matrices}\} \longrightarrow \mathbf{R}$$

with the following defining properties:

1. $\det(I_n) = 1$
2. If we do a *row replacement* on a matrix (add a multiple of one row to another), the determinant does not change.
3. If we *swap two rows* of a matrix, the determinant scales by -1.
4. If we *scale a row* of a matrix by k, the determinant scales by k.

When computing a determinant via row reduction, **try to only use** *row replacement* and *row swaps*. Then you *never have to worry about scaling* by the inverse.

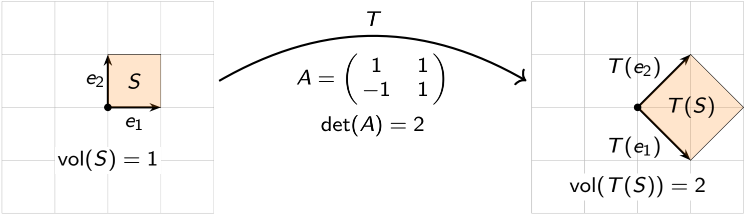
Determinants

Magical properties

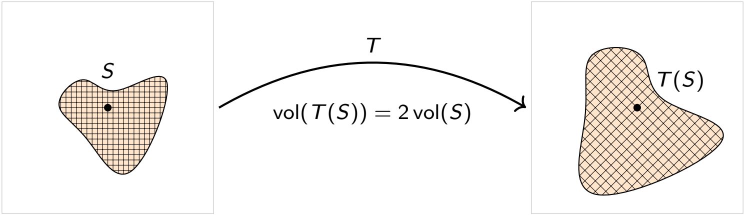
1. There is one and only one function $\det\colon \{\text{square matrices}\} \to \mathbf{R}$ satisfying the defining properties (1)–(4).
2. A is invertible if and only if $\det(A) \neq 0$.
3. If we row reduce A without row scaling, then
$$\det(A) = (-1)^{\#\text{swaps}}\big(\text{product of diagonal entries in REF}\big).$$
4. The determinant can be computed using any of the $2n$ cofactor expansions.
5. $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = \det(A)^{-1}$.
6. $\det(A) = \det(A^T)$.
7. $|\det(A)|$ is the volume of the parallelepiped defined by the columns of A.
8. If A is an $n \times n$ matrix with transformation $T(x) = Ax$, and S is a subset of $\mathbf{R}^n$, then the volume of $T(S)$ is $|\det(A)|$ times the volume of S. (Even for curvy shapes S.)
9. The determinant is multi-linear.

Determinants and Linear Transformations

Why is Property 8 true? For instance, if S is the unit cube, then $T(S)$ is the parallelepiped defined by the columns of A, since the columns of A are $T(e_1), T(e_2), \ldots, T(e_n)$. In this case, Property 8 is the same as Property 7.



For curvy shapes, you break S up into a bunch of tiny cubes. Each one is scaled by $|\det(A)|$; then you use *calculus* to reduce to the previous situation!



Definition

Let A be an $n \times n$ matrix.

1. An **eigenvector** of A is a nonzero vector v in $\mathbf{R}^n$ such that $Av = \lambda v$, for some λ in $\mathbf{R}$. In other words, Av is a multiple of v.
2. An **eigenvalue** of A is a number λ in $\mathbf{R}$ such that the equation $Av = \lambda v$ has a nontrivial solution.

If $Av = \lambda v$ for $v \neq 0$, we say λ is the **eigenvalue for v**, and v is an **eigenvector for λ**.

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The **λ-eigenspace** of A is the set of all eigenvectors of A with eigenvalue λ, plus the zero vector:

$$\begin{aligned} \lambda\text{-eigenspace} &= \{v \text{ in } \mathbf{R}^n \mid Av = \lambda v\} \\ &= \{v \text{ in } \mathbf{R}^n \mid (A - \lambda I)v = 0\} \\ &= \mathrm{Nul}(A - \lambda I). \end{aligned}$$

You find a basis for the λ-eigenspace by finding the parametric vector form for the general solution to $(A - \lambda I)x = 0$ using row reduction.

The Characteristic Polynomial

Definition

Let A be an $n \times n$ matrix. The **characteristic polynomial** of A is

$$f(\lambda) = \det(A - \lambda I).$$

Important Facts:

1. The characteristic polynomial is a polynomial of degree n, of the following form:

$$f(\lambda) = (-1)^n \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0.$$

2. The eigenvalues of A are the roots of $f(\lambda)$.
3. The constant term $f(0) = a_0$ is equal to $\det(A)$:

$$f(0) = \det(A - 0I) = \det(A).$$

4. The characteristic polynomial of a 2×2 matrix A is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A).$$

Definition

The **algebraic multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Similarity

Definition

Two $n \times n$ matrices A and B are **similar** if there is an invertible $n \times n$ matrix P such that

$$A = PBP^{-1}.$$

Important Facts:

1. Similar matrices have the same characteristic polynomial.
2. It follows that similar matrices have the same eigenvalues.
3. If A is similar to B and B is similar to C, then A is similar to C.

Caveats:

1. Matrices with the same characteristic polynomial need not be similar.
2. Similarity has nothing to do with row equivalence.
3. Similar matrices usually do not have the same eigenvectors.

Let $A = PBP^{-1}$, and let $v_1, v_2, \ldots, v_n$ be the columns of P. These form a basis $\mathcal{B}$ for $\mathbf{R}^n$ because P is invertible. *Key relation:* for any vector x in $\mathbf{R}^n$,

$$[Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

This says:

A acts on the usual coordinates of x
in the same way that
B acts on the $\mathcal{B}$-coordinates of x.

Example:

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then $A = PBP^{-1}$. B acts on the usual coordinates by scaling the first coordinate by 2, and the second by $1/2$:

$$B\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ x_2/2 \end{pmatrix}.$$

The unit coordinate vectors are eigenvectors: e_1 has eigenvalue 2, and e_2 has eigenvalue $1/2$.

Similarity
Example

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad [Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

In this case, $\mathcal{B} = \left\{ \binom{1}{1}, \binom{1}{-1} \right\}$. Let $v_1 = \binom{1}{1}$ and $v_2 = \binom{1}{-1}$.

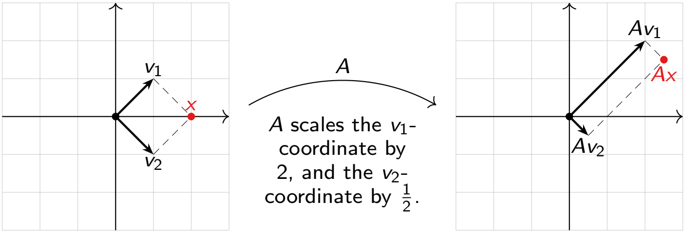
To compute $y = Ax$:

1. Find $[x]_{\mathcal{B}}$.
2. $[y]_{\mathcal{B}} = B[x]_{\mathcal{B}}$.
3. Compute y from $[y]_{\mathcal{B}}$.

Say $x = \binom{2}{0}$.

1. $x = v_1 + v_2$ so $[x]_{\mathcal{B}} = \binom{1}{1}$.
2. $[y]_{\mathcal{B}} = B\binom{1}{1} = \binom{2}{1/2}$.
3. $y = 2v_1 + \frac{1}{2}v_2 = \binom{5/2}{3/2}$.

Picture:



Diagonalization

Definition

An $n \times n$ matrix A is **diagonalizable** if it is similar to a diagonal matrix:

$$A = PDP^{-1} \qquad \text{for } D \text{ diagonal.}$$

It is easy to take powers of diagonalizable matrices:

$$A^n = PD^nP^{-1}.$$

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In this case, $A = PDP^{-1}$ for

$$P = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues (in the same order).

Corollary

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Non-Distinct Eigenvalues

Definition

Let A be a square matrix with eigenvalue λ. The **geometric multiplicity** of λ is the dimension of the λ-eigenspace.

Theorem

Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if, for every eigenvalue λ, the algebraic multiplicity of λ is equal to the geometric multiplicity.

(And all eigenvalues are real, unless you want to diagonalize over $\mathbf{C}$.)

Notes:

- The algebraic and geometric multiplicities are both whole numbers ≥ 1, and the algebraic multiplicity is always greater than or equal to the geometric multiplicity. In particular, they're equal if the algebraic multiplicity is 1.
- Equivalently, A is diagonalizable if and only if the sum of the geometric multiplicities of its eigenvalues is n.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

This has eigenvalues 1 and 2, with algebraic multiplicities 2 and 1, respectively.

The geometric multiplicity of 2 is *automatically* 1.

Let's compute the geometric multiplicity of 1:

$$A - I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This has 1 free variable, so the geometric multiplicity of 1 is 1. This is less than the algebraic multiplicity, so the matrix is *not diagonalizable*.

Applications for Midterm 2

Selected Topics

Production equation

Example

Suppose the maritime sector requires $d = \begin{pmatrix} 20 \\ 35 \\ 80 \end{pmatrix}$: 20,35 and 80 units of production of sectors manufactoring, agriculture and services (MAS), respectively.

How much production x do sectors MAS need to *meet exactly* the demand?

Leontief says

The matrix $(I - C)^{-1}$ *exists* and your **solution** is

$$x = (I - C)^{-1} d$$

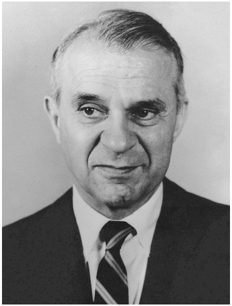
Why?

1. The production itself requires some of the other sectors input: Cx
2. The remaining *(surplus) production* matches exactly the *demand*

$$x = Cx + d$$

We are still assuming that the inverse exists.

Leontief's intuition



When initial demand is d

1. MAS must purchase (from themselves) Cd for the production stage.
2. There is a new order: Cd, which requires its own production stage $C(Cd)$
3. There is a new order: $C^2 d \ldots$

Going out of the loop: At some point new order $C^k d$ is negligible!

Total production is $x \sim d + Cd + C^2 d + \cdots + C^k d = (1 + C + \cdots + C^k)d$

Stochastic Matrices

These arise very commonly in modeling of probabalistic phenomena (Markov chains), where they are also called **transition matrices**.

Some examples:

- Matrices from the population dynamics
- Matrices from the equilibrium-prices economies

Definition

A square matrix A is **stochastic** if all of its entries are nonnegative, and the sum of the entries of each column is 1.

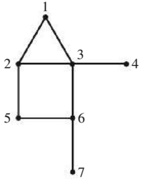
We say A is **regular** if, for some k, all entries of A^k are positive.

Definition

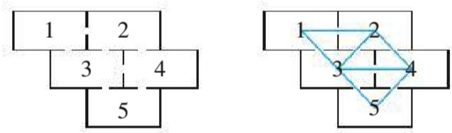
A **steady-state vector** v of A is a non-zero vector with *entries summing to 1* and such that $Av = v$.

Random walks on graphs (a.k.a Mouse on a maze)

A mouse moves freely between rooms/states = selects any with equal probability.



$$
\begin{array}{cc}
\begin{matrix} \quad\; 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} & \\
P = \begin{bmatrix}
0 & 1/3 & 1/4 & 0 & 0 & 0 & 0 \\
1/2 & 0 & 1/4 & 0 & 1/2 & 0 & 0 \\
1/2 & 1/3 & 0 & 1 & 0 & 1/3 & 0 \\
0 & 0 & 1/4 & 0 & 0 & 0 & 0 \\
0 & 1/3 & 0 & 0 & 0 & 1/3 & 0 \\
0 & 0 & 1/4 & 0 & 1/2 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1/3 & 0
\end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix}
\end{array}
$$



$$
\begin{array}{cc}
\begin{matrix} \quad\; 1 & 2 & 3 & 4 & 5 \end{matrix} & \\
P = \begin{bmatrix}
0 & 1/3 & 1/4 & 0 & 0 \\
1/2 & 0 & 1/4 & 1/3 & 0 \\
1/2 & 1/3 & 0 & 1/3 & 1/2 \\
0 & 1/3 & 1/4 & 0 & 1/2 \\
0 & 0 & 1/4 & 1/3 & 0
\end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}
\end{array}
$$

- *Initial state:* The mouse is located at some room i: probabilities

  $v_0 = (x_1, \vdots, x_5)$.
- Probability mouse starts at room 1 is x_1 item *Transition matrix:* $v_{n+1} = Av_n$ means that A dictates how probabilities change.
- Probability mouse is at room 3 after n steps of the walk:

  third entry of v_n.

Non-regular transition matrix

Disconnected states

Consider the following 'transition graph':

The transition matrix is $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$.

Both $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, are eigenvectors with eigenvalue 1.

So there is more than one steady-state vector!

Find the actual Steady State w_1

Red Box example

If one computes $\mathrm{Nul}(A - I)$ and find that $w' = \begin{pmatrix} 7 \\ 6 \\ 5 \end{pmatrix}$

is an eigenvector with eigenvalue 1.

Then, to get a steady state, divide by $18 = 7 + 6 + 5$ to get

$$w = \frac{1}{18} \begin{pmatrix} 7 \\ 6 \\ 5 \end{pmatrix} \sim (0.39, 0.33, 0.28).$$

The long-run

So if you start with 100 total movies, eventually you'll have $100w = (39, 33, 28)$ movies in the three locations, every day.

The time spent on a state

Regardless of the intital location of a particular movie. Eventually, that movie will get 'returned' 39% of the times at location 1, 33% at location 2, and 28% at location 3.

Perron–Frobenius Theorem

These conclusions apply to *any* regular stochastic matrix—whether or not it is diagonalizable!

Perron–Frobenius Theorem

If A is a regular stochastic matrix, then it admits a unique steady state vector w, which spans the 1-eigenspace.

Moreover, for any vector v_0 with entries summing to some number c, the iterates $v_1 = Av_0$, $v_2 = Av_1$, ..., $v_n = Av_{n-1}$, ..., approach cw as n gets large.

Translation:

- The *1-eigenspace* of a regular stochastic matrix A *is a line*.
- The *vector w* has entries that sum to 1, and are *strictly positive*!
- Eventually, the *movies arrange* themselves according to the *steady state percentage*, i.e., $v_n \to cw$.

  (The sum c of the entries of v_0 is the total number of movies)

Google's PageRank: The Importance Rule

Each webpage has an associated importance, or **rank**. This is a positive number.

The Importance Rule

If page P *links* to n other pages $Q_1, Q_2, \ldots, Q_n$, then each Q_i *should inherit* $\frac{1}{n}$ of P's *importance*.

- A very important page links to your webpage: then your webpage is important.
- A ton of unimportant pages link to your webpage: then it's still important.
- But if only one crappy site links to yours, your page isn't important.

Random surfer interpretation

A "random surfer" just randomly clicks on link after link. The pages she *spends the most time* on should be *the most important*. **Stochastic terms:** random walk on the graph of hiperlinks. Look for steady-state vector!

The Google Matrix (Page and Brin's solution)

Fix p in $(0,1)$, called the **damping factor**. (A typical value is $p = 0.15$.)

The **Google Matrix** is

$$M = (1-p)\cdot A + p\cdot B \quad \text{where} \quad B = \frac{1}{N}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

N is the total number of pages, and A is the importance matrix.

- Random surfer interpretation: with probability p the surfer gets bored and starts over on a completely random page.

Fact

The PageRank vector is the steady state for the Google Matrix.

This exists and has positive entries by the Perron–Frobenius theorem.
The hard part is calculating it: the Google matrix has 1 gazillion rows.