MATH 2802
MIDTERM EXAMINATION 1

Name		**Section**	

Please **read all instructions** carefully before beginning.

- There are 6 problems in the exam and the maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- You may use the last page as scratch paper
- Good luck!

1. [2 points each] Circle **T** if the statement is always true and circle **F** if it is ever false.

a) **T** **F** The following augmented matrix corresponds to a system with a unique solution.

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right)$$

b) **T** **F** Any three vectors v_1, v_2, v_3 in $\mathbf{R}^2$ always span $\mathbf{R}^2$.

c) **T** **F** For a transformation $T : \mathbf{R}^4 \to \mathbf{R}^2$, T is never one-to-one.

d) **T** **F** If b is in the span of the columns of A, then $Ax = b$ is consistent.

e) **T** **F** If the columns of an $m \times n$ matrix A form a linearly independent set then $Ax = b$ is consistent for any b in $\mathbf{R}^m$.

Solution.

a) False: The second column represents a free variable, thus there are infinitely many solutions.

b) False: All vectors could lie on the same line through the origin.

c) True: The corresponding matrix will have at least one non-pivot column.

d) True: the span of the columns of A is exactly the set of all b for which $Ax = b$ is consistent.

e) False: If the columns of an $m \times n$ matrix A form a linearly independent set, then the solutions for $Ax = b$ are at most one.

2. Find the matrices corresponding to the following transformations $T, U : \mathbf{R}^2 \to \mathbf{R}^2$.

a) [2pts] T rotates by an angle of 180 degrees.

b) [2pts] U reflects through the $x = y$ line.

c) [4pts] Evaluate $T \circ U(e_1)$ and $T \circ U(e_2)$

Solution.

a) Matrix is $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

b) Matrix is $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

c) U exchanges the axis, then T inverts the sign of the entries. $T \circ U(e_1) = T(e_2) = -e_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and $T \circ U(e_2) = T(e_1) = e_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

3. Consider the following consistent system of linear equations.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= -2 \\ 2x_1 + 4x_2 + 5x_3 + 6x_4 &= -2 \\ -x_1 - 2x_2 - 2x_3 - 2x_4 &= \ \ 0 \end{aligned}$$

a) [4 points] Find the parametric vector form for the general solution.

b) [3 points] Find the parametric vector form of the corresponding *homogeneous* equations.

c) [3 points] Find a linear dependence relation among the vectors

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix} \right\}.$$

Solution.

a) We put the equations into an augmented matrix and row reduce:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 2 & 4 & 5 & 6 & -2 \\ -1 & -2 & -2 & -2 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 2 & -2 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 4 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This means x_2 and x_4 are free, and the general solution is

$$\begin{aligned} x_1 + 2x_2 \quad -2x_4 &= 4 \\ x_3 + 2x_4 &= -2 \end{aligned} \implies \begin{aligned} x_1 &= -2x_2 + 2x_4 + 4 \\ x_2 &= x_2 \\ x_3 &= -2x_4 - 2 \\ x_4 &= x_4 \end{aligned}$$

This gives the parametric vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ -2 \\ 0 \end{pmatrix}.$$

b) Part (a) shows that the solution set of the original equations is the translate of

$$\operatorname{Span}\left\{\begin{pmatrix}-2\\1\\0\\0\end{pmatrix},\begin{pmatrix}2\\0\\-2\\1\end{pmatrix}\right\} \quad \text{by} \quad \begin{pmatrix}4\\0\\-2\\0\end{pmatrix}.$$

We know that the solution set of the homogeneous equations is the parallel plane through the origin, so it is

$$\operatorname{Span}\left\{\begin{pmatrix}-2\\1\\0\\0\end{pmatrix},\begin{pmatrix}2\\0\\-2\\1\end{pmatrix}\right\}.$$

Hence the parametric vector form is

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = x_2\begin{pmatrix}-2\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}2\\0\\-2\\1\end{pmatrix}.$$

c) Solving the vector equation

$$x_1\begin{pmatrix}1\\2\\-1\end{pmatrix} + x_2\begin{pmatrix}2\\4\\-2\end{pmatrix} + x_3\begin{pmatrix}3\\5\\-2\end{pmatrix} + x_4\begin{pmatrix}4\\6\\-2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

amounts to solving the homogeneous system of equations in (b). We have already done so. One nontrivial solution is $x_1 = 2, x_2 = -1, x_3 = 0, x_4 = 0$ (taking $x_2 = -1$ and $x_4 = 0$), so

$$2\begin{pmatrix}1\\2\\-1\end{pmatrix} - 1\begin{pmatrix}2\\4\\-2\end{pmatrix} + 0\begin{pmatrix}3\\5\\-2\end{pmatrix} + 0\begin{pmatrix}4\\6\\-2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Alternatively, one can 'guess from inspection' the coefficients x_1, x_2, x_3 and x_4; then it suffices to verify that these give a linear dependence relation.

4. [7 points] If $A = LU$ with L and U as below; solve the system of equations. $Ax = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(You must use the LU factorization to receive full credit)

Solution.

First, solve the system $L \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$; translating to system of equations we get:

$$\begin{aligned} y_1 \qquad\qquad &= 1 \\ -y_1 + y_2 \qquad &= 0 \\ 2y_1 \qquad + y_3 &= 2 \end{aligned} \implies \begin{aligned} y_1 &= 1 \\ y_2 &= 1 \\ y_3 &= 0 \end{aligned}$$

Second, solve the system $U \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$; translating to the system of equations we get:

$$\begin{aligned} x_1 \qquad + x_3 &= 1 \\ 2x_2 \qquad &= 1 \\ x_3 &= 0 \end{aligned} \implies \begin{aligned} x_1 &= 1 \\ x_2 &= 1/2 \\ x_3 &= 0 \end{aligned}$$

The solution to $Ax = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ is $\begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}$.

5. An apartment building is to be built using a combination of three different floor plans. The arrangement of apartments on each floor is to be chosen from one of the three basic floor plans.

- Each floor of plan A has 3 three-bedroom units, 7 two- bedroom units, and 8 one-bedroom units.
- Each floor of plan B has 4 three-bedroom units, 4 two-bedroom units, and 8 one-bedroom units.
- Each floor of plan C has 5 three-bedroom units, 3 two-bedroom units, and 9 one-bedroom units.

Suppose the building contains a total of x_1 floors of plan A, x_2 floors of plan B, and x_3 floors of plan C.

a) [3pts] What interpretation can be given to the vector $x_1 \begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix}$?

b) [4pts] Write a linear combination of vectors that expresses the total number of three-, two-, and one-bedroom units contained in the building.

c) [3pts] Write a vector equation that represents the following question. (Do not solve the equation)

Is it possible to design the bulding with exactly
66 three-bedroom units, 74 two-bedroom units, and 136 one-bedroom units?

Solution.

a) Vector $x_1 \begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix}$ represents the number of apartments located on floors with plan A.

In other words, the total number of apartments on floors with plan A is $3x_1$ three-bedroom units, $7x_1$ two-bedroom units, and $8x_1$ one-bedroom units.

b) The total number of apartments of each type can be formally written as:

$$x_1 \begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 4 \\ 8 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ 3 \\ 9 \end{pmatrix}$$

c) The linear combination above has to be equated to the target number of apartments of each type:

$$x_1 \begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 4 \\ 8 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 66 \\ 74 \\ 136 \end{pmatrix}$$

6. A car rental in Atlanta has a fleet of 500 cars and two locations, one at the airport and one at Midtown. A car rented at one location may be returned to either of the two locations. The administration has the following information: only 30% of the cars rented at the airport location are returned to the one at Midtown; while 50% of the cars rented at the Midtown location are returned to the airport location.

On Monday there were 300 cars at the airport and 200 cars were at the Midtown location. Let $x_9 = \begin{pmatrix} 300 \\ 200 \end{pmatrix}$ denote the distribution of cars from the rental on February 9th, x_{10} to the distribution of cars from the rental on February 10th, x_{11} to the distribution of cars from the rental on February 11th, and so on.

a) [3pts] Provide the matrix A that estimates $x_{10} = Ax_9$.

b) [1pts] Provide the matrix equation that estimates x_{12} as a product of a power of A and x_9. *(Don't need to justify why)*

c) [1pts] If x_{28} is the distribution of cars from the rental on February 28th, write a matrix equation that estimates x_{28} as a product of a power of A and x_9. *(Don't need to justify why)*

Solution.

a) We have that on vectors $x_9, x_{10}, \ldots$, the first entry corresponds to the number of cars at the airport location, while the second entry corresponds to the number of cars at the Midtown location.

Thus, the first column of A corresponds to the flow 'from airport to': using percentages as values between 0 and 1, the first column of A is $\begin{pmatrix} .7 \\ .3 \end{pmatrix}$. The second column of A corresponds to the flow 'from Midtown to': $\begin{pmatrix} .5 \\ .5 \end{pmatrix}$.

$$A = \begin{pmatrix} .7 & .5 \\ .3 & .5 \end{pmatrix}$$

b) We can use matrix A over and over:

$$x_{12} = Ax_{11} = A(Ax_{10}) = A^2 x_{10} = A^2(Ax_9) = A^3 x_9.$$

c) Following the patern above: $x_{28} = A^{28-9} x_9 = A^{19} x_9$.

(This can be proven by induction)