MATH 2802
MIDTERM EXAMINATION 2

Name		**Section**	

Please **read all instructions** carefully before beginning.

- There are 6 problems in the exam and the maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- You may use the last page as scratch paper
- Good luck!

1. [2 points each] Circle **T** if the statement is always true and circle **F** if it is ever false. The matrices here are $n \times n$.

a) **T** **F** If the geometric multiplicities of eigenvalues in A sum up to n, then A is diagonalizable.

b) **T** **F** A vector $v = (v_1, \dots, v_n)$ is steady-state vector of a stochastic matrix A if $Av = v$ and the length $|v| = \sqrt{v_1^2 + \ldots + v_n^2} = 1$.

c) **T** **F** If P is invertible, then $\det(PDP^{-1}) = \det(D)$.

d) **T** **F** Then dimension of the column space of A is called $rank(A)$.

e) **T** **F** The determinant of an invertible matrix is always positive.

f) **T** **F** A and $B = 2A$ are $n \times n$ matrices. If $\det(A) = 4$, then $\det(B) = 8$.

Solution.

a) True: This means there will exists a basis of $\mathbf{R}^n$ consisting of eigenvectors of A.

b) False: The entries of v should sum to one and be non-negative.

c) True: The determinant of a product equals the product of the determinants of the matrices involved. In addition, $\det(P^{-1}) = \det(P)^{-1}$.

d) True: this is the definition.

e) False: It is non-zero, but the determinant might be negative.

f) False: Matrix B is obtained by scaling each row of A by two. Therefore, the determinant doubles at each of these scalings: $\det(B) = 2^n \det(A) = 4 \cdot 2^n$.

2. [10 points]
Consider the decomposition of $A = PDP^{-1}$ with

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix}$$

a) Draw both the 1-eigenspace and the 1/3-eigenspace of A.

b) Provide an eigenvector of A with eigenvalue 1.

c) Evaluate $A^{100}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

d) Write a formula for D^n.

Solution.

a) [4pts] Sketch both the 1-eigenspace and the 1/3-eigenspace of A.

b) [2pts] Vector corresponding to first column of P, or any scalar multiple of it.

c) [2pts] Since $\binom{1}{0}$ is an eigenvector with eigenvalue 1, no matter how many times we multiply by A, the result is always $\binom{1}{0}$. That is

$$A^{100}\begin{pmatrix}1\\0\end{pmatrix} = A^{99}\begin{pmatrix}1\\0\end{pmatrix} = \ldots = A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}.$$

d) [2pts] $D^n = \begin{pmatrix}1 & 0\\0 & 3^{-n}\end{pmatrix}.$

3. [8 points] In this problem, show your work and justify your answers.

a) Is 5 an eigenvalue of $A = \begin{pmatrix} -2 & 7 \\ -5 & 10 \end{pmatrix}$?

b) Is $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ an eigenvector of $B = \begin{pmatrix} 7 & 6 & -1 \\ 0 & 4 & 8 \\ 3 & -8 & 17 \end{pmatrix}$?

c) Find the 3-eigenspace of $C = \begin{pmatrix} 4 & -7 & 0 & 2 \\ 0 & 3 & -4 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

d) Is C diagonalizable?

Solution.

a) [2 points] We compute $\det(A-\lambda I) = (-2-\lambda)(10-\lambda)+35 = \lambda^2-8\lambda+15$. We can see that $\lambda = 5$ is a root of this polynomial, thus 5 is, indeed, an eigenvalue of A.

b) [1 points] By computing $Bv = \begin{pmatrix} 12 \\ 12 \\ 12 \end{pmatrix}$ we can see that v is eigenvector of B with eigenvalue 12.

c) [3 points] We have to find the null space of $C - 3I$. That is, find the parametric vector form of solutions to $(C - 3I)x = 0$.

$$C - 3I = \begin{pmatrix} 1 & -7 & 0 & 2 \\ 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -7 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The system translates to $x_1 = 7x_2$ and $x_3 = x_4 = 0$. In other words, solutions have the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 7 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and the 3-eigenspace is $Span \left\{ \begin{pmatrix} 7 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

d) [2 points] The matrix is not diagonalizable because the algebraic multiplicity of 3 is strictly larger than its geometric multiplicity.

4. [10 points] Consider a matrix A which is row-equivalent to:

$$A = \begin{pmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 8 & 0 & 5 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Find a basis for $Col(A)$

b) What is the dimension for $Nul(A)$?

c) Justify your answer in b) *(There are several possible correct answers)*

d) The ____________ theorem states that if the dimension of V is m then:

- Any m linearly independent vectors in V form ____________ for V .
- Any m vectors that ____________ form ____________ for V .

Solution.

a) [3pts] A basis is found by colecting the columns in A corresponding to the pivot columns in its row-reduction:

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} -3 \\ 3 \\ 5 \\ -5 \end{pmatrix} \right\}$$

b) [1pt] Dimension of $Nul(A)$ is 2.

c) [1pt] Since A has 5 columns and the dimension of $Col(A)$ is 3, by the Rank theorem, the dimension of $Nul(A) = 5 - 3 = 2$. Alternatively, we can see that A has two non-pivot columns, so the basis of $Nul(A)$ will consist of two vectors.

d) [3pts] If the dimension of V is m then:

- Any m **linearly independent** vectors in V form a **basis** for V .
- Any m vectors that **span V** form a **basis** for V .

5. [5pts]

There is a car rental with locations at the airport, Midtown and Marietta. After a comprehensive study the administration knows that:

- A car rented at the airport has 20% chance of being returned to the Midtown location and 10% chance of being returned to the Marietta location.
- A car rented at the Midtown location has 10% chance of being returned to the airport and 10% chance of being returned to the Marietta location.
- A car rented at the Marietta location has 30% chance of being returned to the airport and 30% chance of being returned to the Midtown location.

a) Find the transition matrix Q (make transitions from airport, midtown and Marietta correspond to columns 1,2 and 3 respectively).

b) If the steady-state vector of Q is $w = \frac{1}{28}\begin{pmatrix} 9 \\ 15 \\ 4 \end{pmatrix}$. What percentage of time will a car in the rental be returned to the Midtown location?

Solution.

a) [3 points] $Q = \begin{pmatrix} .7 & .1 & .3 \\ .2 & .8 & .3 \\ .1 & .1 & .4 \end{pmatrix}$

b) [2 points] The percentage corresponds to the second entry: fraction is $\frac{15}{28} \sim 54\%$

6. [7 points] An economy of coal and electric sectors has production matrix $C = \begin{pmatrix} 0 & .5 \\ .6 & .2 \end{pmatrix}$ and a demand of $d = \begin{pmatrix} 50 \\ 30 \end{pmatrix}$ is requested. Use Leontief's inverse matrix to determine the production level x necessary to satisfy the demand d.

Hint: remember that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Solution.

[2pts] The inverse matrix is $A = (I - C)^{-1}$

[2pts] The production level is $x = (I - C)^{-1}d$

[2pts] Since $\det(A) = .5$, the inverse is

$$\begin{pmatrix} 1 & -.5 \\ -.6 & .8 \end{pmatrix}^{-1} = \frac{1}{det(A)}\begin{pmatrix} .8 & .5 \\ .6 & 1 \end{pmatrix} = \begin{pmatrix} 1.6 & 1 \\ 1.2 & 2 \end{pmatrix}$$

[1pt] Then $x = \begin{pmatrix} 1.6 & 1 \\ 1.2 & 2 \end{pmatrix}\begin{pmatrix} 50 \\ 30 \end{pmatrix} = \begin{pmatrix} 110 \\ 120 \end{pmatrix}$.