MATH 2802
MIDTERM EXAMINATION 3

Name		**Section**	

Please **read all instructions** carefully before beginning.

- There are 6 problems in the exam and the maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- You may use the last page as scratch paper
- Good luck!

1. [2 points each] Circle **T** if the statement is always true and circle **F** if it is ever false. The matrices here are $n \times n$.

a) **T** **F** If z is orthogonal to both v and w, then z is in the orthogonal complement of $Span\{v, w\}$.

b) **T** **F** The orthogonal projection of y onto W depends on the basis selected for W.

c) **T** **F** If $\{v_1, v_2, v_3\}$ is an orthogonal basis of W then $\{v_1, v_2, 5v_3\}$ is an orthogonal basis for W.

d) **T** **F** If P is orthonormal then $P^{-1} = P^T$

e) **T** **F** The quadratic form Q is positive definite if for all non-zero vectors $Q(x) \geq 0$.

Solution.

a) True: For any vector $c_1 v + c_2 w$ we have $z \cdot (c_1 v + c_2 w) = 0$; that is, z is orthogonal to all vectors in $Span\{v, w\}$.

b) False.

c) True. To verify this, compute the dot product between pairs in $\{v_1, v_2, 5v_3\}$.

d) True.

e) False: The inequality should be strict; that is, $Q(x) > 0$ for all non-zero vectors.

2. [6 points] Let $W = Span\{v_1, v_2\}$ with $v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Find $y = proj_W \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$. Justify your answer.

Solution.

[2points] The justification: v_1 is orthogonal to v_2; that is, $v_1 \cdot v_2 = (-1)(1) + 0(1) + (1)(1) = 0$ so we can use formulas from class.

[2points] Thus, we use the next formula to find the projection of $x = \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$ onto W:

$$[2pts] proj_W \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix} = \frac{x \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{x \cdot v_2}{v_2 \cdot v_2} v_2 = \frac{6}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \frac{9}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}.$$

3. [6 points] Let $v_1 = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}, v_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

a) Compute $v_1 \cdot v_2$, $v_2 \cdot v_3$ and $v_1 \cdot v_3$ (Show your work).

b) Is $\{v_1, v_2, v_3\}$ a basis of $\mathbf{R}^3$? Justify your answer.

Solution.

a) [4 points] $v_1 \cdot v_2 = v_1 \cdot v_3 = v_2 \cdot v_3 = 0$

b) [2 points] We know that any set of orthogonal vectors is linearly independent; since there are three vectors, then the basis theorem guarantees they are a basis of $\mathbf{R}^3$.

4. [8pts]

Find the best approximation to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ by vectors of the form $c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. What is the error in your approximation?

Solution.

Let $W = Span \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$; this subspace is generated by orthogonal vectors (so we can use the formulas from class).

[4points] The best approximation to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is

$$proj_W \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

[2points] which has the form $1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

[2points] The error in the approximation is the distance between $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$:

$$\left| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right| = \left| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right| = 1.$$

5. [10 points]

Apply the Gram-Schmidt procedure to the vectors $v_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, v_2 = \begin{pmatrix}0\\1\\1\\1\end{pmatrix}$ and $v_3 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$

and give an orthonormal basis for $Span\{v_1, v_2, v_3\}$.

Solution.

The Gram-Schmidt procedure will give vectors w_1, w_2, w_3:

a) [1points] $w_1 = v_1$

b) [2points] $w_2 = v_2 - proj_{w_1}(v_2)w_1 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1$

c) [1points]

$$w_2 = \begin{pmatrix}0\\1\\1\\1\end{pmatrix} - \frac{3}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}-3/4\\1/4\\1/4\\1/4\end{pmatrix}$$

d) [2points] $w_3 = v_3 - proj_{w_1}(v_3)w_1 - proj_{w_2}(v_3)w_2 = v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2$

e) [1points]

$$w_3 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix} - \frac{2}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{(2/4)}{(12/16)}\begin{pmatrix}-3/4\\1/4\\1/4\\1/4\end{pmatrix} = \frac{1}{6}\left[\begin{pmatrix}0\\0\\6\\6\end{pmatrix} - \begin{pmatrix}3\\3\\3\\3\end{pmatrix} - \begin{pmatrix}-3\\1\\1\\1\end{pmatrix}\right] = \frac{1}{3}\begin{pmatrix}0\\-2\\1\\1\end{pmatrix}$$

f) [3points] Now we divide each vector w_1, w_2, w_3 by its length, to obtain a set of orthogonal vectors with unit length:

$$\left\{\frac{1}{\sqrt{4}}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{12}}\begin{pmatrix}-3\\1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}0\\-2\\1\\1\end{pmatrix}\right\}$$

6. [10 points] Let $Q(x_1,x_2) = 3x_1^2 + 2x_1x_2 + 3x_2^2$. There is a matrix P such that the change of variables $y = P^{-1}x$ gives the following form $Q(y) = ay_1^2 + by_2^2$.

a) Find the matrix associated to $Q(x) = x^TAx$

b) Find an orthogonal decomposition for $A = PDP^T$

c) Give the values of a and b

d) Is the quadratic form $Q(x)$ indefinite? Justify your answer.

Solution.

a) [1points] $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

b) Find the orthogonal decomposition $A = PDP^T$:

[2points] Characteristic polynomial, factorized:

$$\det(A - \lambda I) = (3-\lambda)^2 - 1 = (\lambda - 4)(\lambda - 2)$$

So eigenvalues of A are 4 and 2. Remains to find corresponding eigenvectors, and normalize them

[1 points] For $\lambda = 4$, one eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

[1 points] For $\lambda = 2$, one eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

[3 points] Write down $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$.

c) [1 point] Values of $a = 4$ and $b = 2$

d) [1points] No. It is positive definite because all eigenvalues of A are positive.