Section 1.3-1.4

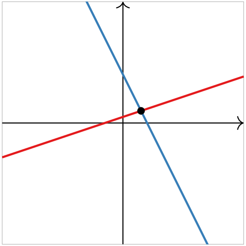
Vector and Matrix Equations

Motivation

Linear algebra's *two viewpoints*:

- **Algebra**: systems of equations and their solution sets
- **Geometry**: intersections of points, lines, planes, etc.

$$\begin{aligned} x - 3y &= -3 \\ 2x + y &= 8 \end{aligned} \quad \rightsquigarrow$$



The **geometry** will give us *better insight into the properties* of systems of equations and their solution sets.

Most Important Today: Spans and Solutions to Equations

Let $\mathbf{b} \in \mathbf{R}^n$ and A be a matrix with columns $v_1, v_2, \ldots, v_n \in \mathbf{R}^n$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

Very Important Fact

$Ax = b$ has a solution

$\iff$ b is a linear combination of $v_1, \ldots, v_n$

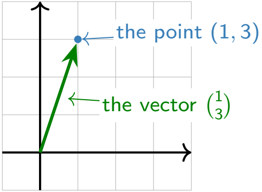
$\iff$ b is in the span of the columns of A.

The last condition is geometric.

Vectors

Elements of $\mathbf{R}^n$ can be considered *points*...

or **vectors**:
arrows with a given
length and direction.

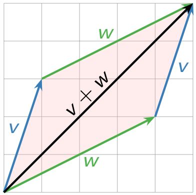


It is *convenient* to express **vectors** in $\mathbf{R}^n$ **as matrices** with n rows and *one column*:

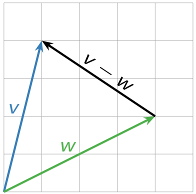
$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Addition (*parallelogram law*) and Substraction

Addition: Is the one that commutes



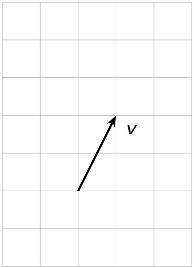
Substraction: If you add **v − w** to **w**, you get **v**.



Towards "linear spaces"

Scalar multiples of a vector:
have the same *direction* but a different *length*.
The scalar multiples of **v** *form a line*.

Some multiples of v.



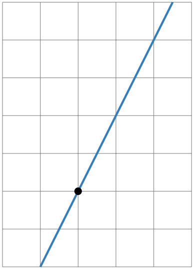
$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$2\mathbf{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$-\frac{1}{2}\mathbf{v} = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}$$

$$0\mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

All multiples of v.



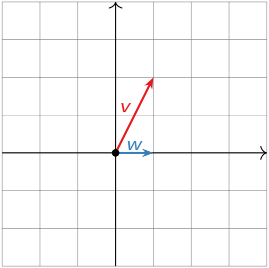
Linear Combinations: *generate new vectors*

Definition

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

We call $\mathbf{w}$ a **linear combination** of the vectors $v_1, v_2, \ldots, v_p$, and the scalars $c_1, c_2, \ldots, c_p$ are called the *weights* or *coefficients*.

Example



Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

What are some linear combinations of v and w?

- $v + w$
- $v - w$
- $2v + 0w$
- $2w$
- $-v$

Matrix × Vector

Let A be an $m \times n$ matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad \text{with columns } v_1, v_2, \ldots, v_n$$

Definition

The **product** of A with a vector x in $\mathbf{R}^n$ is the linear combination

$$Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \stackrel{\text{def}}{=} x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

- Necessary: Number of columns of A equals number of rows of x.
- The output is a vector in $\mathbf{R}^m$.

Matrix × Vector

Another way

A *row vector* is a matrix with one row.

Dot product

The **product** of a row vector of length n and a (column) vector of length n

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \stackrel{\text{def}}{=} a_1x_1 + \cdots + a_nx_n.$$

is a scalar!

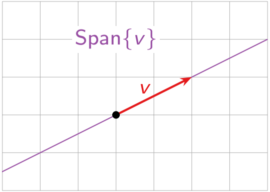
If A is an $m \times n$ matrix with rows $r_1, r_2, \ldots, r_m$, and x is a vector in $\mathbf{R}^n$, then

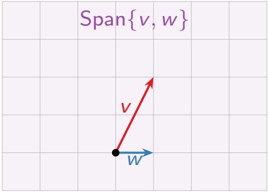
$$Ax = \begin{pmatrix} \text{—}\, r_1 \,\text{—} \\ \text{—}\, r_2 \,\text{—} \\ \vdots \\ \text{—}\, r_m \,\text{—} \end{pmatrix} x = \begin{pmatrix} r_1x \\ r_2x \\ \vdots \\ r_mx \end{pmatrix}$$

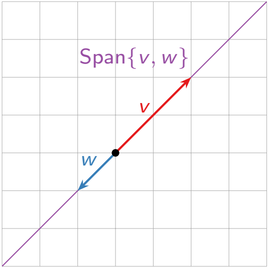
This is **a vector in $\mathbf{R}^m$**.

Pictures of Span in R^2

Drawing a picture of Span$\{v_1, v_2, \ldots, v_p\}$ is the same as drawing a picture of all linear combinations of $v_1, v_2, \ldots, v_p$.







Span

Definition

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted $\text{Span}\{v_1, v_2, \ldots, v_p\}$. In symbols:

In other words:

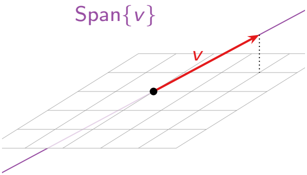
- $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is the subset **spanned/generated by** $v_1, v_2, \ldots, v_p$.
- it's exactly the *collection of all b in* $\mathbf{R}^n$ such that the *vector equation*

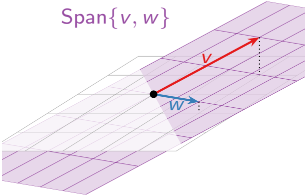
$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

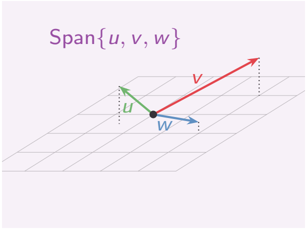
is consistent i.e., has a solution (unknowns $x_1, x_2, \ldots, x_p$).

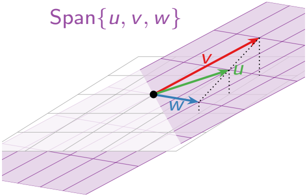
Poll

Pictures of Span in $\mathbf{R}^3$









Systems of Linear Equations

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3 \end{aligned} \quad \overset{\text{matrix form}}{\rightsquigarrow} \quad \left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right)$$

$$\overset{\text{row reduce}}{\rightsquigarrow} \quad \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array}\right)$$

$$\overset{\text{solution}}{\rightsquigarrow} \quad \begin{aligned} x &= -1 \\ y &= -9 \end{aligned}$$

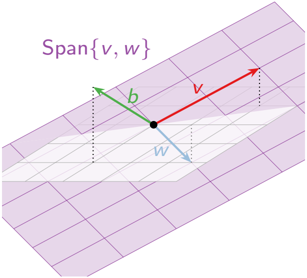
Conclusion:

$$-\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - 9\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

Systems of linear equations depend on the Span of a set of vectors!

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?



Columns of A:

$$\color{red}{v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}} \qquad \color{blue}{w = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$$

Output vector:

$$\color{green}{b = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}}$$

Is b contained in the span of the columns of A?

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?

Answer: Let's check by solving the matrix equation using row reduction.

The first step is to put the system into an augmented matrix.

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

The last equation is $0 = 1$, so the system is *inconsistent*.

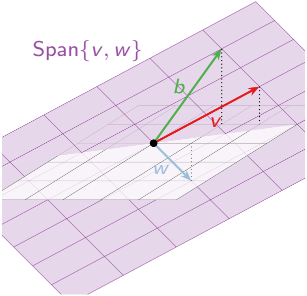
In other words, the matrix equation

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

has no solution.

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?



Columns of A:

$$v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad w = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Solution vector:

$$b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Is b contained in the span of the columns of A?

Spans and Solutions to Equations

Example 3, explained

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?

Answer: Let's do this systematically using row reduction.

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

This gives us

$$x = 1 \qquad y = -1.$$

This is consistent with the picture on the previous slide:

$$1\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \qquad \text{or} \qquad A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Linear Systems, Vector Equations, Matrix Equations, . . .

Now have four equivalent ways of writing linear systems:

1. As a *system of equations*:

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5 \end{aligned}$$

2. As an *augmented matrix*:

$$\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right)$$

3. As a *vector equation* ($x_1 v_1 + \cdots + x_n v_n = b$):

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a *matrix equation* ($Ax = b$):

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

We will move back and forth freely between these over and over again.

When Solutions Always Exist

Equivalent means that, for any given list of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{b}$, *either all three* statements are true, *or all three* statements are false.

Theorem

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{b}$ be vectors in $\mathbf{R}^n$ and $x_1, x_2, \ldots, x_p$ be scalars. The following **are equivalent**

1. A vector $\mathbf{b}$ is in the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.
2. The vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$, has a solution.
3. The augmented matrix below corresponds to a consistent linear system

$$\left(\begin{array}{cccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p & \mathbf{b} \end{array} \right).$$

Theorem

Let A be an $m \times n$ (non-augmented) matrix. The following are **equivalent**

1. $Ax = b$ has a *solution for all b* in $\mathbf{R}^m$.
2. The span of the columns of A is *all of $\mathbf{R}^m$*.
3. A has a pivot *in each row*.

Why is (1) the same as (3)?

Look at **reduced echelon** forms of A.

- If A has *a pivot in each row*:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{pmatrix} \quad \text{and } (\,A \mid b\,) \text{ reduces to:} \quad \left(\begin{array}{ccccc|c} 1 & 0 & \star & 0 & \star & \star \\ 0 & 1 & \star & 0 & \star & \star \\ 0 & 0 & 0 & 1 & \star & \star \end{array}\right).$$

  There's no b that makes it inconsistent, so there's *always a solution*.

- If A *doesn't have a pivot* in each row:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and this can be made inconsistent:} \quad \left(\begin{array}{ccccc|c} 1 & 0 & \star & 0 & \star & 0 \\ 0 & 1 & \star & 0 & \star & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{array}\right).$$