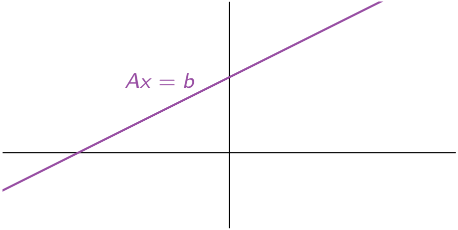
Section 1.5

Solution Sets of Linear Systems

First Part Today

Describe and draw the solution set of $Ax = b$, using spans and parametric vector solutions.



Recall: the **solution set** is the collection of all vectors x such that $Ax = b$ is true.

Solution in Parametric vector form

Question

What is the solution set of $Ax = 0$, where $A =$

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equations}} \begin{cases} x_1 \phantom{{}+x_2} - 8x_3 - 7x_4 = 0 \\ x_2 + 4x_3 + 3x_4 = 0 \end{cases}$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 8x_3 + 7x_4 \\ x_2 = -4x_3 - 3x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

Let A be an $m \times n$ matrix. If the *free variables* in the equation $Ax = b$ are $x_i, x_j, x_k, \ldots$

And the **parametric vector form** of the solution is

$$x = b' + x_i v_i + x_j v_j + x_k v_k + \cdots$$

for some vectors $b', v_i, v_j, v_k, \ldots$ in $\mathbf{R}^n$, and any scalars $x_i, x_j, x_k, \ldots$

Then the solution set is

$$b' \quad + \quad \mathrm{Span}\{v_i,\ v_j,\ v_k, \ldots\}.$$

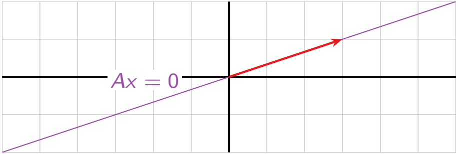
Homogeneous Example

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for any x_2 in $\mathbf{R}$. The solution set is $\mathrm{Span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$.



Note: *one* free variable means the solution set is a *line* in $\mathbf{R}^2$ ($2 = \#$ variables $= \#$ columns).

Inhomogeneous Example

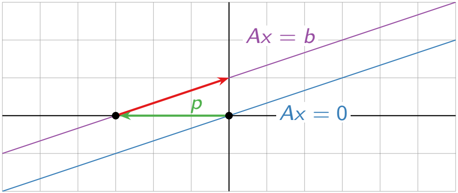
Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ for any x_2 in $\mathbf{R}$.

This is a *translate* of $\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$: it is the parallel line through $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$.



It can be written

$$\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

Homogeneous Systems

Definition

A system of linear equations of the form $Ax = 0$ is called *homogeneous.*
A system of linear equations of the form $Ax = b$ with $b \neq 0$ is called **nonhomogeneous** or **inhomogeneous.**

A homogeneous system *always has the solution* $x = 0$. This is called the *trivial solution.* The nonzero solutions are called **nontrivial.**

Observation

$Ax = 0$ has a *nontrivial* solution

$\iff$ there is a free variable

$\iff$ A has a *column with no pivot*.

Poll

Poll

How many solutions can there be to a homogeneous system with more equations than variables?

A. 0

B. 1

C. ∞

The trivial solution is always a solution to a homogeneous system, so answer A is impossible.

This matrix has only one solution to $Ax = 0$:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

This matrix has infinitely many solutions to $Ax = 0$:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Solutions for Homogeneous Systems

If u and v are solutions to $Ax = 0$, then so is every vector in $\mathrm{Span}\{u, v\}$. **Why?**

$$\begin{cases} Au = 0 \\ Av = 0 \end{cases} \implies A(c_1 u + c_2 v) = c_1 Au + c_2 Av = c_1 0 + c_2 0 = 0.$$

(Here 0 means the zero vector.)

Important

The set of **solutions to $Ax = 0$** is **a span**.

Solutions for **Consistent** Nonhomogeneous Systems

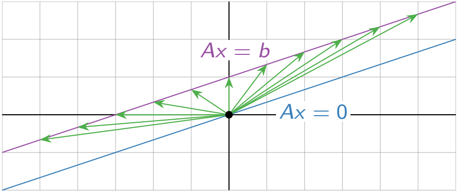
When consistent

The set of solutions to $Ax = b$, is **parallel to a span**.

Solutions are obtained by taking one **particular solution** p to $Ax = b$, and *adding all solutions to $Ax = 0$*.

Why? If $Ap = b$ and $Ax = 0$, then $p + x$ is also a solution to $Ax = b$:

$$A(p + x) = Ap + Ax = b + 0 = b.$$



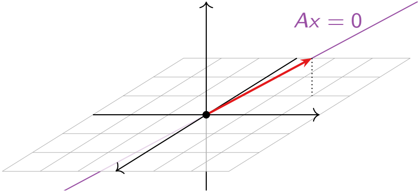
Note:
Works for *any specific solution p*: it doesn't matter how one found it!

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix}?$$

Answer: Span $\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}$.



Note: *one* free variable means the solution set is a *line* in $\mathbf{R}^3$ ($3 = \#$ variables $= \#$ columns).

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equations}} \begin{cases} x_1 \phantom{{}+x_2} - 2x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 2x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \end{cases}$$

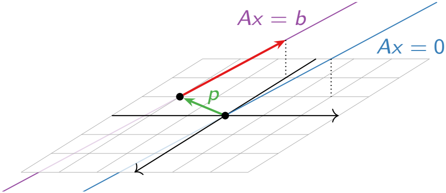
$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -5 \\ -3 \\ -2 \end{pmatrix}?$$

Answer: $\operatorname{Span}\left\{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}.$



The solution set is a *translate* of

$$\operatorname{Span}\left\{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right\}:$$

it is the parallel line through

$$p = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}.$$

A Nonhomogeneous System

Example, explained

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -5 \\ -3 \\ -2 \end{pmatrix}?$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & -5 \\ 2 & -1 & -5 & -3 \\ 1 & 0 & -2 & -2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & 0 & -2 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\text{equations}} \begin{cases} x_1 \phantom{{}+x_2} - 2x_3 = -2 \\ \phantom{x_1 - {}} x_2 + x_3 = -1 \end{cases}$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 2x_3 - 2 \\ x_2 = -x_3 - 1 \\ x_3 = x_3 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}.$$

Second Part today

Review Section 1.6 with *MyMathLab*