Tuesday, January 23

Office Hours

Jun Tao Duan	- Thursdays 4-6pm	Clough 280
Scott Guan	- Tuesdays 2-3pm	Skiles 230
Sophia Dever	- Tuesdays 4-5pm	Clough Math Lab

Math is fun!

Undergrads can apply for TAing in the math department:

http://www.math.gatech.edu/undergraduate-ta

Section 1.9 (& 1.8)

The Matrix of a Linear Transformation

Unit Coordinate Vectors

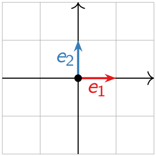
Definition

The **unit coordinate vectors** in $\mathbf{R}^n$ are

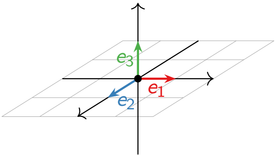
This is what $e_1, e_2, \ldots$ mean, *for the rest of the class.*

$$e_1 = \begin{pmatrix}1\\0\\\vdots\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\\vdots\\0\\0\end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix}0\\0\\\vdots\\1\\0\end{pmatrix}, \quad e_n = \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}.$$

in $\mathbf{R}^2$



in $\mathbf{R}^3$



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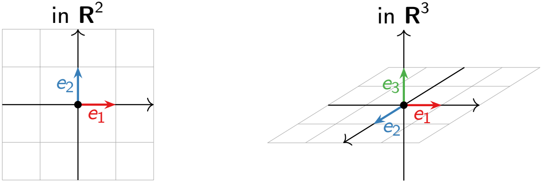

Important: if A is an $m \times n$ matrix with *columns* $v_1, v_2, \ldots, v_n$, then $Ae_i = v_i$ for $i = 1, 2, \ldots, n$: the transformation $T(x) = Ax$ sends e_i *to vector* v_i.

Linear Transformations are Matrix Transformations

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a **linear transformation**. Let

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

This is an _____ matrix,

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$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

This is an $m \times n$ matrix, and T is the matrix transformation for A: $T(x) = Ax$.

The matrix A is called the *standard matrix for T*.

Take-Away

A linear transformation *may not be given* a priori
as a matrix transformation
but **linear** transformations are **the same as matrix** transformations.

Matrix Transformations

Projection

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^3 \to \mathbf{R}^3$. Then

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

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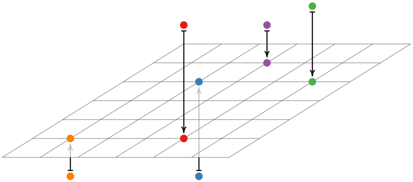
This is *projection onto the xy-axis*. Picture:



Matrix Transformations

Reflection

Let $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^2$. Then

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}.$$

Matrix Transformations

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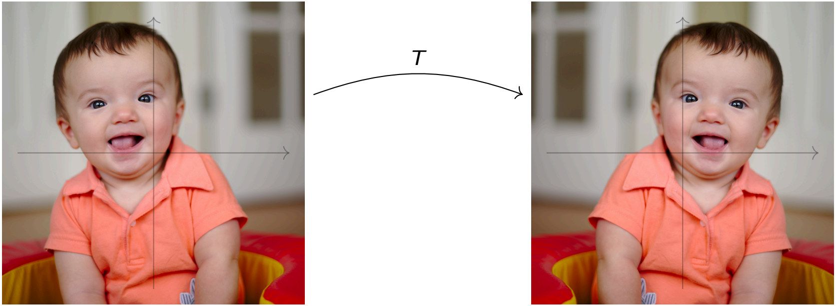
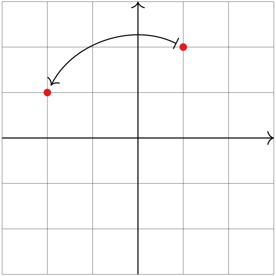
This is *reflection over the y-axis*. Picture:



Define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$. Is T linear?

This is called **rotation** (by 90°). Picture:

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$



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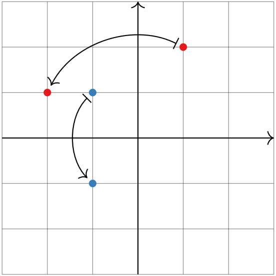

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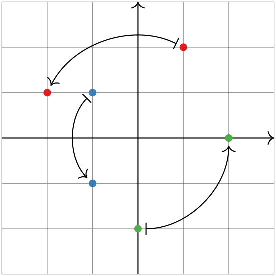
$$T\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

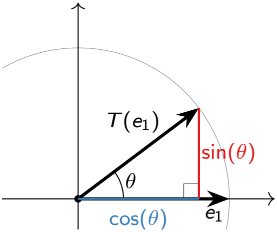


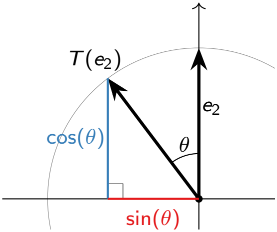
Linear Transformations: Rotation

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined by

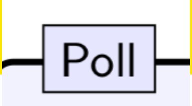
$$T(x) = x \text{ rotated counterclockwise by an angle } \theta?$$





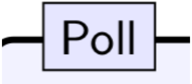
Linear Transformations: Reflexion/Projection

Construction Phase 1



Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?



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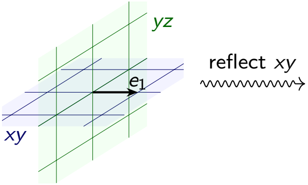

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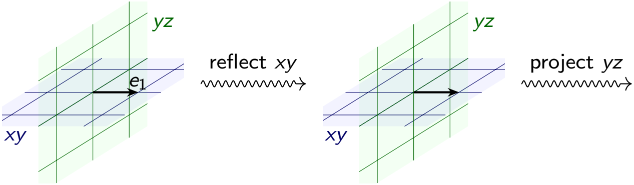

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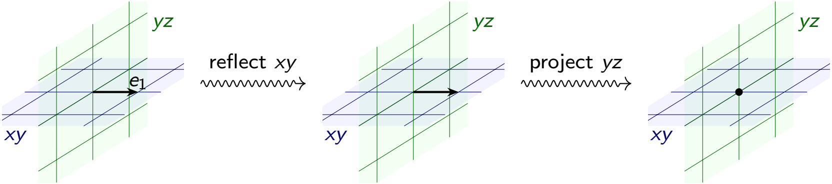

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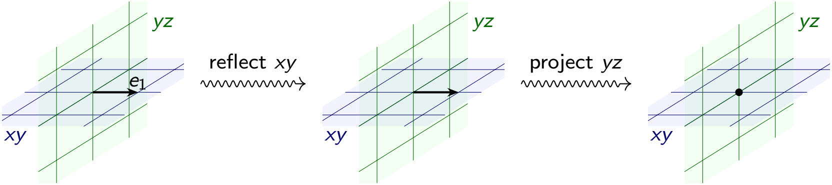

$$T(e_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

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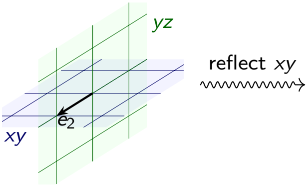

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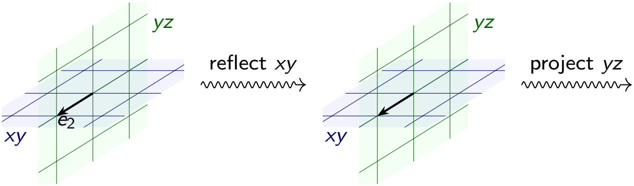

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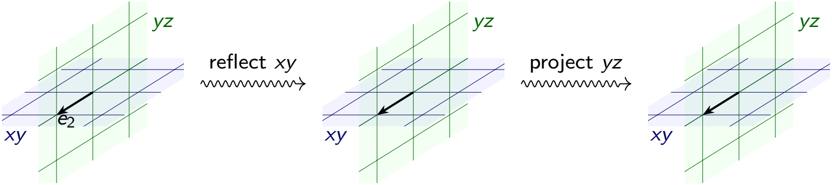

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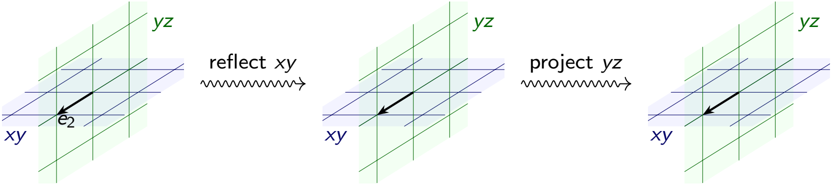

$$T(e_2) = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

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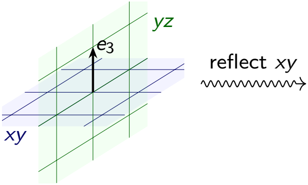

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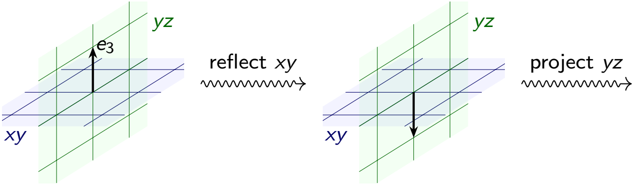

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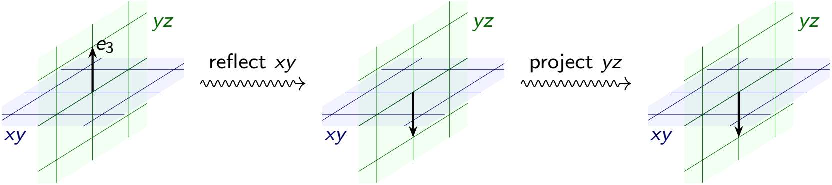

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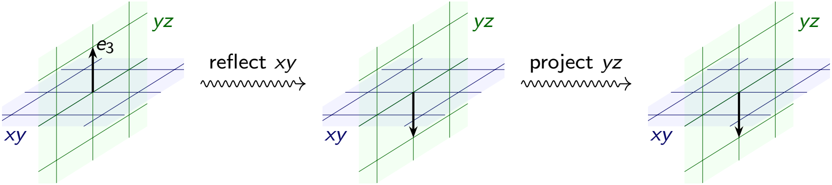

$$T(e_3) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

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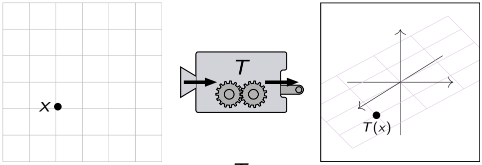
Transformations

Definition

A **transformation** (or **function** or **map**) from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule T that assigns to each vector x *in* $\mathbf{R}^n$ a vector $T(x)$ *in* $\mathbf{R}^m$.

Notation:

$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m$ means T is a transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$.



$$\mathbf{R}^n \xrightarrow{\quad T \quad} \mathbf{R}^m$$

domain codomain

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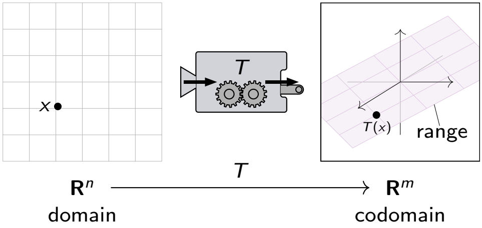

Think of T as a *"machine"*

- takes x as an input
- *gives you* $T(x)$ as the output.

Linear Transformations

Recall: If A is a matrix, u, v are vectors, and c is a scalar, then

$$A(u+v) = Au + Av \qquad A(cv) = cAv.$$

So if $T(x) = Ax$ is a matrix transformation then,

$$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cu) = cT(u)$$

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if it satisfies the above equations for *all vectors* u, v in $\mathbf{R}^n$ and *all scalars* c.

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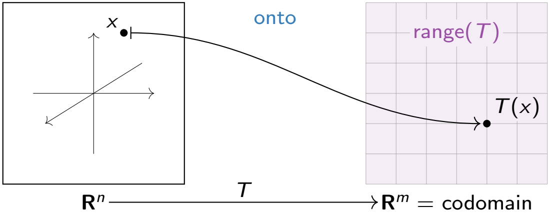
In other words, T **"respects" addition and scalar multiplication**.

Onto Transformations

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **onto** (or **surjective**) if the *range of T* is equal to $\mathbf{R}^m$ (its codomain). In other words, *each b in $\mathbf{R}^m$ is the image of at least one x* in $\mathbf{R}^n$:



Characterization of Onto Transformations

Theorem

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A. *Then the following are equivalent:*

- T **is onto**

Characterization of Onto Transformations

Theorem

Let $T: \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A. *Then the following are equivalent:*

- **T is onto**
- $T(x) = b$ has a solution for every b in $\mathbf{R}^m$
- $Ax = b$ is *consistent for every b* in $\mathbf{R}^m$
- A has a pivot in every row
- **The columns of A span $\mathbf{R}^m$**

One-to-one Transformations

Definition

A transformation $T \colon \mathbf{R}^n \to \mathbf{R}^m$ is **one-to-one** (or **into**, or **injective**) if *different vectors* in $\mathbf{R}^n$ *map to different vectors* in $\mathbf{R}^m$.

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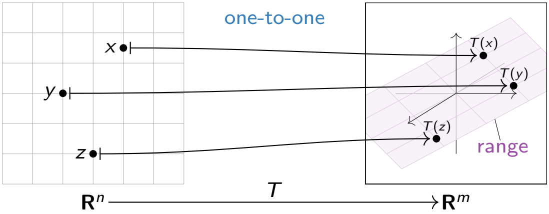

Characterization of One-to-One Transformations

Theorem

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A. *Then the following are equivalent*:

- **T is one-to-one**

Characterization of One-to-One Transformations

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A. *Then the following are equivalent*:

- **T is one-to-one**
- $T(x) = b$ has one or zero solutions for every b in $\mathbf{R}^m$
- $Ax = b$ has a *unique solution or is inconsistent* for every b in $\mathbf{R}^m$
- $Ax = 0$ has a unique solution
- A has a pivot in every ______.