# Announcements

Tuesday, January 30

- Quizzes will be handed back in recitation
- **Solution to quizzes** can be found in the calendar:

  http://people.math.gatech.edu/~leslava3/1718S-2802/schedule.html
- *Diversity and Inclusion Project:* Accent stories

  http://www.diversity.gatech.edu/DIFellowsProgram/2017ProjectSpotlights/

Towards a *'flipped classroom'*:

- Homework and quizzes cover the material from same week
- Don't wait until lecture to start learning the material

# Sections 1.7, 2.2,2.3

Invertibility: Criterions and and algorithm

## Motivation

We can turn any system of linear equations into a matrix equation

$$Ax = b.$$

*This notation is suggestive*. Can we solve the equation by **"dividing by A"**?

$$x \overset{??}{=} \frac{b}{A}$$

Answer: Only sometimes.

Today: **Invertibility**: "dividing" by a matrix = *multiplying* by the inverse

So when does the inverse of a matrix exists?

# The Definition of Inverse

## Definition

Let $A$ be an $n \times n$ square matrix. We say $A$ is **invertible** (or **nonsingular**) if there is a matrix $B$ of the same size, such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

identity matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

In this case, $B$ is the **inverse** of $A$, and is *written* $A^{-1}$.

## Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Wild guess: $B = A^{-1}$. Check:

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

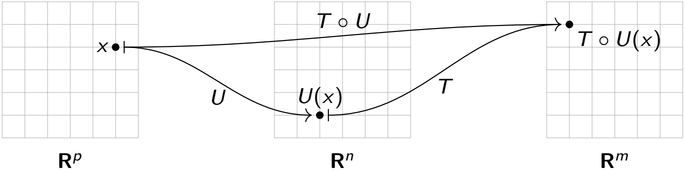
$$BA = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

✔

## Definition

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ and $U\colon \mathbf{R}^p \to \mathbf{R}^n$ be transformations. The **composition** is the transformation

$$T \circ U\colon \mathbf{R}^p \to \mathbf{R}^m \quad \text{defined by} \quad T \circ U(x) = T(U(x)).$$




A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is **invertible** if there exists $U\colon \mathbf{R}^n \to \mathbf{R}^n$ such that for all $x$ in $\mathbf{R}^n$

$$T \circ U(x) = x \quad \text{and} \quad U \circ T(x) = x.$$

In this case we say $U$ is the **inverse** of $T$, and we **write** $U = T^{-1}$.

In other words, $T(U(x)) = x$, so $T$ *"undoes"* $U$, and likewise $U$ "undoes" $T$.

# Invertible Transformations

**Fact**

A transformation $T$ is invertible if and only if *it is both one-to-one and onto*.

This means *for every $y$* in $\mathbf{R}^n$, *there is a unique $x$* in $\mathbf{R}^n$ such that $T(x) = y$.

Therefore we can define $T^{-1}(y) = x$.

**Important**

- The matrix of the composition is the product of the matrices!
- The product of invertible matrices is invertible:
  The *inverse is the product* of the inverses, in the *reverse order*.

# Caveats of Matrix Multiplication

Beware: matrix multiplication is very subtle:

- $AB$ is *usually not equal* to $BA$.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

  In fact, $AB$ may be defined when $BA$ is not.

- *No cancellation*: $AB = AC$ does not imply $B = C$, even if $A \neq 0$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$$

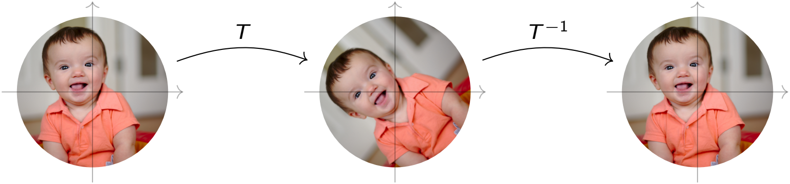
- *Not necessarily zero matrices*: $AB = 0$ does not imply $A = 0$ or $B = 0$ .

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

# Invertible Transformations

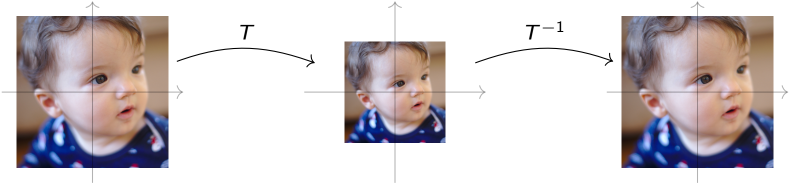
## Examples

Let $T$ = **counterclockwise** rotation in the plane by $45^\circ$. What is $T^{-1}$?




$T^{-1}$ is *clockwise* rotation by $45^\circ$.

Let $T$ = **shrinking** by a factor of $2/3$ in the plane. What is $T^{-1}$?

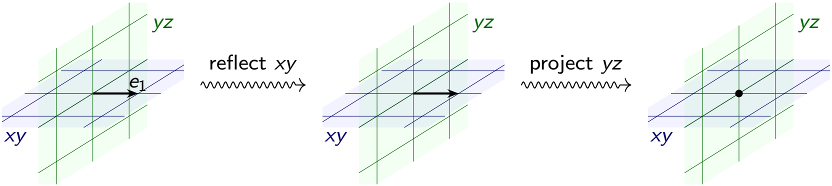



$T^{-1}$ is *stretching* by $3/2$.

**How to** find the inverse of a matrix in general?

## Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the $xy$-plane and then projects onto the $yz$-plane?




1. $U_1$ reflects through the $xy$-plane: $$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

2. $U_2$ projects ono the $yz$-plane: $$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. $T = U_2 \circ U_1$: $$B = A_2 A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

# Elementary Matrices

## Definition

An **elementary matrix** is a matrix $E$ that *differs* from $I_n$ by one *row operation*.

There are **three kinds**, corresponding to the three elementary row operations:

scaling ($R_2 = 2R_2$)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

row replacement ($R_2 = R_2 + 2R_1$)

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

swap ($R_1 \longleftrightarrow R_2$)

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Important Fact: For any $n \times n$ matrix $A$, if $E$ is the elementary matrix for a row operation, then *EA differs from A by the same row operation*.

Example:

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} \xrightarrow{R_2 = R_2 + 2R_1} \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 10 \\ 0 & -3 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 10 \\ 0 & -3 & -4 \end{pmatrix}$$

# Poll

Elementary matrices are invertible. The inverse is the elementary matrix which un-does the row operation.

**Poll**

Let $E$ be the $3 \times 3$ matrix corresponding to *adding 2 times row 3 to row 2*. Mark $E^{-1}$ from the list below

$$a) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{pmatrix} \qquad b) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

$$c) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \qquad d) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution: $E^{-1}$ is equal to $c$).

# Computing $A^{-1}$

Let $A$ be an $n \times n$ matrix. Here's how to compute $A^{-1}$.

1. Row reduce the augmented matrix $( A \mid I_n )$.
2. If the result has the form $( I_n \mid B )$, then $A$ is invertible and $B = A^{-1}$.
3. Otherwise, $A$ is not invertible.

Example

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}$$

# Computing $A^{-1}$

## Example

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 = R_3 + 3R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 = R_2 - R_3]{R_1 = R_1 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 \div 2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 3/2 & 1/2 \end{array}\right)$$

So $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{pmatrix}$.

Check: $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix} \begin{pmatrix} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✔

# Why Does This Work?

First answer: We can think of the algorithm as *simultaneously solving* the equations

$$Ax_1 = e_1 : \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$Ax_2 = e_2 : \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$Ax_3 = e_3 : \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

- From theory: $x_i = A^{-1}Ax_i = A^{-1}e_i$. So *$x_i$ is the i-th column of $A^{-1}$*.
- *Row reduction*: the solution *$x_i$ appears* in $i$-th column in the augmented part.

Second answer: Through *elementary matrices* (important for next class)

# Why Does The Inversion Algorithm Work?

## Theorem

An $n \times n$ matrix $A$ is invertible if and only if it is row equivalent to $I_n$.

Why? Say the row operations taking $A$ to $I_n$ are the elementary matrices $E_1, E_2, \ldots, E_k$. So

pay attention to the order! $\longrightarrow$

$$\begin{aligned} E_k E_{k-1} \cdots E_2 E_1 A &= I_n \\ \implies E_k E_{k-1} \cdots E_2 E_1 A A^{-1} &= A^{-1} \\ \implies E_k E_{k-1} \cdots E_2 E_1 I_n &= A^{-1}. \end{aligned}$$

This is what we do when row reducing the augmented matrix:
*Do same row operations* to $A$ (first line above) and to $I_n$ (last line above).
Therefore, you'll end up with $I_n$ and $A^{-1}$.

$$\left( A \mid I_n \right) \rightsquigarrow \left( I_n \mid A^{-1} \right)$$

# The Really Big Theorem for Square Matrices of Math 2802

## The Invertible Matrix Theorem

Let $A$ **be an** $n \times n$ **matrix**, and let $T \colon \mathbf{R}^n \to \mathbf{R}^n$ be defined by $T(x) = Ax$.
*The following statements are equivalent*.

1. $A$ is invertible.
2. $T$ is invertible.
3. $T$ is one-to-one.
4. $T$ is onto.
5. $A$ has a left inverse (there exists $B$ such that $BA = I_n$).
6. $A$ has a right inverse (there exists $B$ such that $AB = I_n$).
7. $A^T$ is invertible.
8. $A$ is row equivalent to $I_n$.
9. $A$ has $n$ pivots (one on each column and row).
10. The columns of $A$ are linearly independent.
11. $Ax = 0$ has only the trivial solution.
12. The columns of $A$ span $\mathbf{R}^n$.
13. $Ax = b$ is consistent for all $b$ in $\mathbf{R}^n$.

you really have to understand these

# Approach to The Invertible Matrix Theorem

As with all Equivalence theorems:

- For **invertible matrices**: all statements of the Invertible Matrix Theorem are true.
- For **non-invertible matrices**: *all statements* of the Invertible Matrix Theorem *are false*.

**Tackle the assertions!**

**You know enough** at this point to be able to *reduce all* of the statements *to assertions about the pivots* of a square matrix.

Strong recommendation: If you want to understand invertible matrices, go through all of the conditions of the IMT and *try to figure out on your own* why they're all equivalent.

# Linear Independence and Matrix Columns

By definition, $\{v_1, v_2, \ldots, v_p\}$ is *linearly independent* if and only if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution. This holds *if and only if* the matrix equation

$$Ax = 0$$

has only the trivial solution, where $A$ is the *matrix with columns* $v_1, v_2, \ldots, v_p$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

This is true if and only if the matrix $A$ has *a pivot in each column*.

**Important**

- The vectors $v_1, v_2, \ldots, v_p$ are linearly independent if and only if the matrix with columns $v_1, v_2, \ldots, v_p$ has a pivot in each column.
- Solving the matrix equation $Ax = 0$ will either verify that the columns $v_1, v_2, \ldots, v_p$ of $A$ are linearly independent, or will produce a linear dependence relation.

# Linear Independence: Two more facts

Fact 1: Say $v_1, v_2, \ldots, v_n$ are in $\mathbf{R}^m$. *If $n > m$* then $\{v_1, v_2, \ldots, v_n\}$ is *linearly dependent*: the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

cannot have a pivot in each column (it is too wide).

This says you can't have 4 linearly independent vectors in $\mathbf{R}^3$, for instance.

> A wide matrix can't have linearly independent columns.

Fact 2: *If one* of $v_1, v_2, \ldots, v_n$ *is zero*, then $\{v_1, v_2, \ldots, v_n\}$ is *linearly dependent*. For instance, if $v_1 = 0$, then

$$1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + \cdots + 0 \cdot v_n = 0$$

is a linear dependence relation.

> A set containing the zero vector is linearly dependent.

# Linear Independence

## (Algorithmic) increasing span criterion

If the vector $v_j$ **is not in** Span$\{v_1, v_2, \ldots, v_{j-1}\}$,

it means Span$\{v_1, v_2, \ldots, v_j\}$ *is bigger* than Span$\{v_1, v_2, \ldots, v_{j-1}\}$.

**If true for all $j$**

A set of vectors is linearly independent if and only if, every time *you add another vector* to the set, the *span gets bigger*.

### Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is **linearly independent** if and only if, *for every $j$*, the *span of $v_1, v_2, \ldots, v_j$ is strictly larger* than the span of $v_1, v_2, \ldots, v_{j-1}$.

Suppose a set of vectors $\{v_1, v_2, \ldots, v_p\}$ is *linearly dependent*.

Take the **largest** $j$ such that $v_j$ *is in the span* of the others.

Is $v_j$ is in the *span of* $v_1, v_2, \ldots, v_{j-1}$?

For example, $j = 3$ and

$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$$

Rearrange:

$$v_4 = -\frac{1}{6}\left(2v_1 - \frac{1}{2}v_2 - v_3\right)$$

so $v_4$ is also in the span of $v_1, v_2, v_3$, but $v_3$ was supposed to be the last one that was in the span of the others.