Section 2.5

LU Factorization

First factorization of MATH 2802

A guru provides, for (suitable) $m \times n$ matrix A, matrices L and U such that

- L is lower triangular $m \times m$ matrix (with *ones on the diagonal*)
- U is an $m \times n$ row echelon form (not necessary reduced)
- $A = LU$

E.g.

$$A = \begin{pmatrix} 2 & 4 & -1 & 5 & -1 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

The matrix LU

It helps to interpret L as *instructions on how to sum rows* of U.

A short cut to solving equations

How to save time in solving $Ax = b$?

1. Visit the guru and *get L and U*,
2. *Quickily* solve $Ly = b$,
3. *Quickily* solve $Ux = y$,
4. Claim that x **is a solution** to $Ax = b$.

Are we **allowed to** do that?

Check if x is really a solution: Start with the guru's information $A = LU$

$$Ax = LUx = L(Ux) = Ly = b$$

Interpret

Transformation A as *a composition*: first *apply* transformation U then L.
Trace back to find a solution: First solve for L, then solve for U.

Efficiency

Dense matrices -few entries are zeros-
Sparse matrices -most entries are zeros.

Save up to 2/3 of the time

If $n \geq 30$, A is $n \times n$ and dense

- Computing L and U: $\sim 2n^3/3$ flops.
- Solving a triangular system: $\sim n^2$ flops

vs.

- Computing A^{-1}: $\sim 2n^3$ flops
- Matrix multiplication $A^{-1}x$: $\sim 2n^2$
- *Extra downside:* roundoff errors in A^{-1}

Avoid dense matrices

If A is sparse it is likely that

- A^{-1} is dense
- L and U are sparse

Become the guru and sell factorizations

Our assumption: Row reduction of A requires *no row sawps*

1. *Row reduce* the matrix A, but *do not normalize pivots* to being one's; e.g.

$$(E_6 E_5 E_4 E_3 E_2 E_1)A = U$$

2. Separate the row reduction according to 'clearing' pivot columns

$$(E_6)(E_5 E_4)(E_3 E_2 E_1)A = U$$

3. Now left multiply by the inverses of the elementary matrices:

$$A = ((E_6)(E_5 E_4)(E_3 E_2 E_1))^{-1} U = LU$$

Looks like a very complicated matrix:

$$L = ((E_6)(E_5 E_4)(E_3 E_2 E_1))^{-1}$$

but can be **read off the row-reduction steps**.

Construct the matrix L

$$L = ((E_6)(E_5E_4)(E_3E_2E_1))^{-1}$$

2. Separate the row reduction according to 'clearing' pivot columns

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} = A_1$$

$$\sim A_2 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U$$

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix}$$

$$\div 2 \quad \div 3 \quad \div 2 \quad \div 5$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

Construct the matrix L

$$L = ((E_6)(E_5E_4)(E_3E_2E_1))^{-1}$$

2. Separate the row reduction according to 'clearing' pivot columns

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} = A_1$$

$$\sim A_2 = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} = U$$

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 5 \end{bmatrix}$$

$$\div 2 \qquad \div 3 \qquad \div 2 \qquad \div 5$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\begin{bmatrix} 1 & & & \\ -2 & 1 & & \\ 1 & -3 & 1 & \\ -3 & 4 & 2 & 1 \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

Process on the backstage

This backstage process can be *read off the row-reduction steps*.

1. *Divide into streaks* of elementary matrices with non-zero entries on the same column. E.g. $E_3 E_2 E_1$
2. *Superposition:* The product of these elementary matrices will have the same non-entries (see atomic triangular matrix).
3. *Inverse:* $(E_3 E_2 E_1)^{-1}$ changes the sign of all off-diagonal entries.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix}$$

$$(E_3 E_2 E_1)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix}$$

A second example

Find the LU factorization of A:

$$A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & -3 & -1 & 6 \\ 0 & 6 & 2 & -7 \\ 0 & -9 & -3 & 13 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 10 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

$$\begin{bmatrix} 2 \\ 6 \\ 2 \\ 4 \\ -6 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -3 \\ 6 \\ -9 \end{bmatrix} \quad \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix}$$

$$\div 2 \qquad \div 3 \qquad \div 5$$

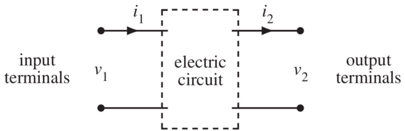
$$\begin{bmatrix} 1 & & & \\ 3 & 1 & & \\ 1 & -1 & 1 & \cdots \\ 2 & 2 & -1 & \\ -3 & -3 & 2 & \end{bmatrix},$$

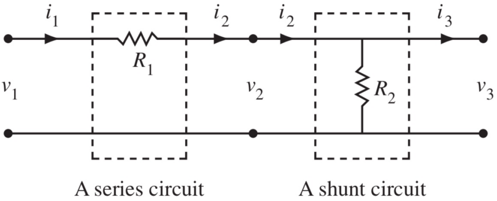
$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & -1 & 1 & 0 \\ -3 & -3 & 2 & 0 & 1 \end{bmatrix}$$

Electrical networks

Electric networks can be analysed as composition of smaller networks.

$$\begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = A \begin{pmatrix} v_1 \\ i_1 \end{pmatrix}$$





Transfer matrix *A*: relation between the *input* $\begin{pmatrix} v_2 \\ i_2 \end{pmatrix}$ and the *output* $\begin{pmatrix} v_1 \\ i_1 \end{pmatrix}$.

Engineers gives the following information:

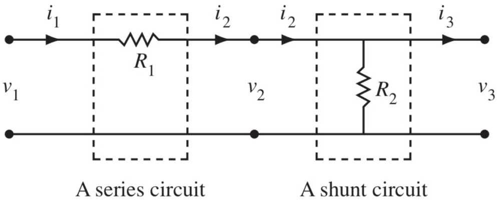
$$\begin{bmatrix} 1 & -R_1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ -1/R_2 & 1 \end{bmatrix}$$

Transfer matrix of series circuit

Transfer matrix of shunt circuit

A ladder circuit

If $\begin{pmatrix} v_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the resistances in the circuit are given by $R_1 = 3$ and $R_2 = .5$. Give the value of $\begin{pmatrix} v_3 \\ i_3 \end{pmatrix}$



$$\begin{pmatrix} v_3 \\ i_3 \end{pmatrix} = A_2 \begin{pmatrix} v_2 \\ i_2 \end{pmatrix} = A_2 A_1 \begin{pmatrix} v_1 \\ i_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$