Section 2.6

The Leontief model

The cost-production matrix

- Column corresponding to sector s shows *where* the production of *s is required* among all sectors.

$$C = \begin{pmatrix} .0 & .4 & .6 \\ .6 & .1 & .2 \\ .4 & .5 & .2 \end{pmatrix}$$

Distribution of Output from:			
Coal	**Electric**	**Steel**	**Purchased by:**
.0	.4	.6	Coal
.6	.1	.2	Electric
.4	.5	.2	Steel

E.g.

- Coal industry sells all its production to electric and steel industry.
- To procude electricity, the plant needs 10% of its production to run.
- To produce steel elements, we need to use 20% of steel production to build tools and factory.

Read off cost-production matrix

- If total production of coal pays x_1 dollars, then the **revenue of coal from the distinct sectors** is given by

$$C\begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = x_1\begin{pmatrix} 0 \\ .4 \\ .6 \end{pmatrix}$$

  That is, 40% *of revenue* is paid by electric sector
  and 60% *of revenue* is paid by steel sector

- If coal, electric and steel sectors set **prices of production** at x_1, x_2 and x_3,

$$Cx = \begin{pmatrix} .4x_2 + .6x_3 \\ .6x_1 + .1x_2 + .2x_3 \\ .4x_1 + .5x_2 + .2x_3 \end{pmatrix}$$

  then *expenses for coal* industry are given by 'coal' *entry of vector Cx*:

$$.4x_2 + .6x_3$$

Economy equilibrium prices

- **Equilibrium prices** can be assigned to the total outputs of the various sectors in such a way that the income of each sector balances its expenses

If coal, electric and steel sectors set **prices of production** at x_1, x_2 and x_3,

$$Cx = \begin{pmatrix} .4x_2 + .6x_3 \\ .6x_1 + .1x_2 + .2x_3 \\ .4x_1 + .5x_2 + .2x_3 \end{pmatrix}$$

then expenses *for coal* industry are equal to total revenue of coal sector:

$$x_1 = .4x_2 + .6x_3$$

Find equilibrium prices

Solve the linear equation $Cx = x$; equivalently solve

$$(I - C)x = 0$$

- This is a *non-profit* economy.

Input-Output model

When an order arrives

- To receive orders, sectors **must have surplus production** to sell.
- That is, entries per column *add to less than 1*

$$C = \begin{pmatrix} .5 & .4 & .2 \\ .2 & .3 & .1 \\ .1 & .1 & .3 \end{pmatrix}$$

	Inputs Consumed per Unit of Output		
Purchased from:	**Manufacturing**	**Agriculture**	**Services**
Manufacturing	.50	.40	.20
Agriculture	.20	.30	.10
Services	.10	.10	.30

If manufacturing sector sells 100 units it needs to buy from other sectors:

$$C \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} = 100 \begin{pmatrix} .5 \\ .2 \\ .1 \end{pmatrix} = \begin{pmatrix} 50 \\ 20 \\ 10 \end{pmatrix}$$

E.g. To produce tennis shoes, manufacturing sector needs to buy laces (from manufacturing sector)

Production equation

Example

Suppose the maritime sector requires $d = \begin{pmatrix} 20 \\ 35 \\ 80 \end{pmatrix}$: 20,35 and 80 units of production of sectors manufactoring, agriculture and services (MAS), respectively.

How much production x do sectors MAS need to *meet exactly* the demand?

Leontief says

The matrix $(I - C)^{-1}$ *exists* and your **solution** is

$$x = (I - C)^{-1} d$$

Why?

1. The production itself requires some of the other sectors input: Cx
2. The remaining *(surplus) production* matches exactly the *demand*

$$x = Cx + d$$

We are still assuming that the inverse exists.

Existence of $(I - C)^{-1}$

Real numbers exercise: Prove that $\sum_{k=0}^{\infty} x^k$ is finite when $|x| < 1$.

- In fact $\sum_{k=0}^{\infty} x^k = (1-x)^{-1}$

$$(1-x)(1+x+x^2) = 1 - x^3$$
$$(1-x)(1+x+x^3) = 1 - x^4$$
$$(1-x)(1+x+\ldots+x^k) = 1 - x^{k+1}$$

Proof: Since x^{k+1} is smaller and smaller ($x^k \to 0$):

$$(1-x)\sum_{k=0}^{\infty} x^k = 1$$

So the infinite sum and $1 - x$ are **inverse of each other**.

Fact

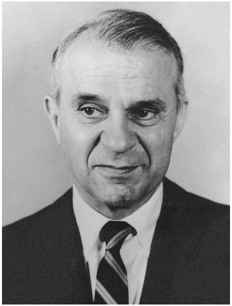
For a matrix C, if entries are positive and its *column sums* are less than 1, then

$$\lim_{k\to\infty} C^k = 0.$$

(because all eigenvalues of C are less than 1)

- $(1-C)^{-1}$ exists and equals $I + C + C^2 + C^3 \cdots$

Leontief's intuition



When initial demand is d

1. MAS must purchase (from themselves) Cd for the production stage.
2. There is a new order: Cd, which requires its own production stage $C(Cd)$
3. There is a new order: $C^2 d \ldots$

Going out of the loop: At some point new order $C^k d$ is negligible!

Total production is $x \sim d + Cd + C^2 d + \cdots + C^k d = (1 + C + \cdots + C^k)d$