Announcements

Tuesday, February 13

Good job on the midterm!!

- This week is a review from MATH 1553
- Comprehensive notes can be found in `http://people.math.gatech.edu/~leslava3/1718S-2802.html`
- Selected material at `http://people.math.gatech.edu/~leslava3/1718S-2802/schedule.html`

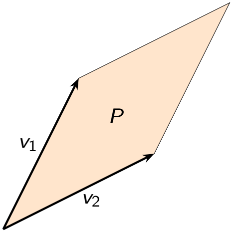
Sections 3.1-3.2

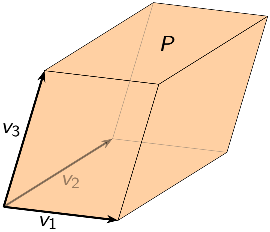
Determinants

The Idea of Determinants

Let A be an $n \times n$ matrix. Determinants are only for square matrices.

The columns $v_1, v_2, \ldots, v_n$ give you n vectors in $\mathbf{R}^n$. These determine a **parallelepiped** P.





Observation: the volume of P is zero $\iff$ the columns are *linearly dependent* (P is "flat") $\iff$ the matrix A is not invertible.

The **determinant** of A will be a number $\det(A)$ whose absolute value is the *volume of P*.
In particular, $\det(A) \neq 0 \iff A$ is invertible.

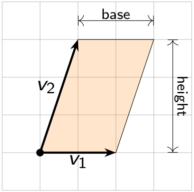
Determinants of 2×2 Matrices

Revisited

There is a formula in the 2×2 case:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

What does this have to do with volumes?



$$v_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The area of the parallelogram is

$$\text{base} \times \text{height} = 2 \cdot 3 = \left| \det \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \right|.$$

The area of the parallelogram is always $|ad - bc|$. If v_1 is not on the x-axis: it's a fun geometry problem!

Note: The volume is zero if and only if the columns are collinear

Determinants of 3×3 Matrices

Here's a formula:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

How to remember this?

Draw a bigger matrix, repeating the first two columns to the right:

$$+ \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix}$$

For example,

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} = \begin{vmatrix} 5 & 1 & 0 & 5 & 1 \\ -1 & 3 & 2 & -1 & 3 \\ 4 & 0 & -1 & 4 & 0 \end{vmatrix} = -15 + 8 + 0 - 0 - 0 - 1 = -8$$

Cofactor expansion

Recursive formula: you compute a larger determinant *in terms of smaller ones*.

Let A be an $n \times n$ matrix.

$$\begin{aligned} A_{ij} &= ij\text{th } \textbf{minor} \text{ of } A \\ &= (n-1) \times (n-1) \text{ matrix you get by } \textit{deleting the } i\text{th row and } j\text{th column} \end{aligned}$$

$$C_{ij} = ij\text{th } \textbf{cofactor} \text{ of } A = (-1)^{i+j} \det A_{ij}$$

The signs of the cofactors follow a *checkerboard pattern*:

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix} \qquad \pm \text{ in the } ij \text{ entry is the sign of } C_{ij}$$

Definition

The **determinant** of an $n \times n$ matrix A is

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}.$$

This formula is called **cofactor expansion** *along the first row*.

Example: Cofactor expansion along first row

$$\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} = 5\cdot\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & -3 & 2 \\ 4 & 0 & -1 \end{pmatrix} - 1\cdot\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

$$+ 0\cdot\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

$$= 5\cdot\det\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} - 1\cdot\det\begin{pmatrix} -1 & 2 \\ 4 & -1 \end{pmatrix} + 0\cdot\det\begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix}$$

$$= 5\cdot(-3-0) - 1\cdot(1-8)$$

$$= -15 + 7 = -8$$

Cofactor expasion: Specify point of reference...

Recall: the cofactor expansion *along the first row.*

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

Actually, you can expand cofactors along any row or column you like! Good

trick: Use cofactor expansion along a row or a column *with a lot of zeros*.

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

It looks easiest to expand along the third column:

$$\det A = 0 \cdot \det \begin{pmatrix} \text{don't} \\ \text{care} \end{pmatrix} - 0 \cdot \det \begin{pmatrix} \text{don't} \\ \text{care} \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = 2 - 1 = 1$$

Poll

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$$\det\begin{pmatrix} 1 & 7 & -5 & 14 & 3 & 22 \\ 0 & -2 & -3 & 13 & 11 & 1 \\ 0 & 0 & -1 & -9 & 7 & 18 \\ 0 & 0 & 0 & 3 & 6 & -8 \\ 0 & 0 & 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} = ?$$

A. -6 B. -3 C. -2 D. -1 E. 1 F. 2 G. 3 H. 6

If you *expand repeatedly along the first column*, you get

$$1\cdot\det\begin{pmatrix} -2 & -3 & 13 & 11 & 1 \\ 0 & -1 & -9 & 7 & 18 \\ 0 & 0 & 3 & 6 & -8 \\ 0 & 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = 1\cdot(-2)\cdot\det\begin{pmatrix} -1 & -9 & 7 & -18 \\ 0 & 3 & 6 & -8 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= 1\cdot(-2)\cdot(-1)\cdot\det\begin{pmatrix} 3 & 6 & -8 \\ 0 & 1 & -11 \\ 0 & 0 & -1 \end{pmatrix} = 1\cdot(-2)\cdot(-1)\cdot 3\cdot\det\begin{pmatrix} 1 & -11 \\ 0 & -1 \end{pmatrix}$$

$$= 1\cdot(-2)\cdot(-1)\cdot 3\cdot 1\cdot(-1) = -6$$

The Determinant of an Upper-Triangular Matrix

Theorem

The determinant of an upper-triangular matrix is the product of the diagonal entries:

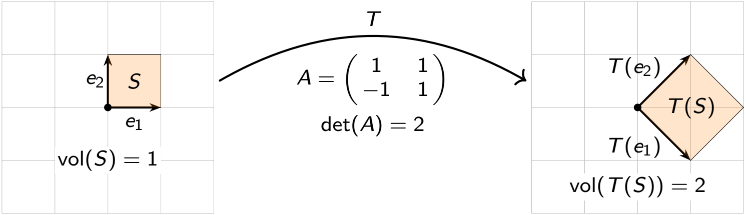
$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} = a_{11}a_{22}a_{33}\cdots a_{nn}.$$

Trick: Expand along the last row

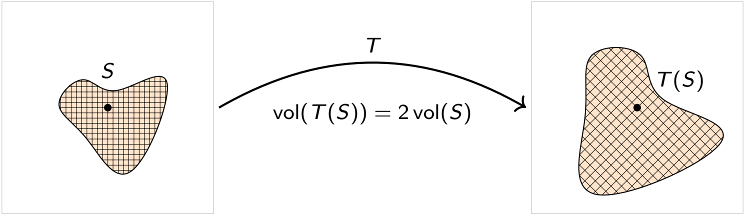
This works for any matrix that is *upper-triangular* (all entries below the main diagonal are zero).

The same is true for lower-triangular matrices. (Repeatedly expand along the first row.)

If S is the *unit cube*, then $T(S)$ is the parallelepiped formed by the columns of A. The **volumen changes** according to $\det(A)$.



For curvy regions: break S up into *tiny cubes*; each one is scaled by $|\det(A)|$. Then use *calculus* to reduce to the previous situation!



Defining the Determinant in Terms of its Properties

Definition

The **determinant** is a function

$$\det\colon \{\text{square matrices}\} \longrightarrow \mathbf{R}$$

with the following defining properties:

1. $\det(I_n) = 1$
2. If we do a *row replacement* on a matrix, the determinant does not change.
3. If we *swap* two rows of a matrix, the determinant scales by -1.
4. If we *scale a row* of a matrix by k, the determinant scales by k.

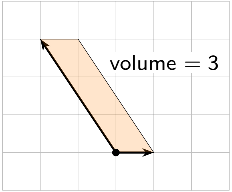
Why would we think of these properties? This is how volumes work!

1. The volume of the *unit cube* is 1.
2. Volumes don't change under *a shear*.
3. Volume of a *mirror image* is negative of the volume?
4. If you *scale one coordinate* by k, the volume is multiplied by k.

Properties of the Determinant

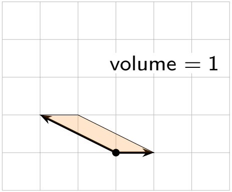
2×2 matrix

$$\det \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = 3$$



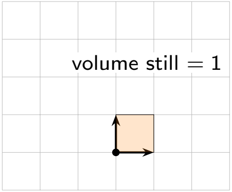
Scale: $R_2 = \frac{1}{3} R_2$

$$\det \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = 1$$



Row replacement: $R_1 = R_1 + 2R_2$

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$



Magical Properties of the Determinant

you really have to know these

1. det: {**square matrices**} $\to$ **R** is *the only function* satisfying the defining properties (1)–(4).

2. A is invertible if and only if $\det(A) \neq 0$.

3. If we row reduce A without row scaling, then

$$\det(A) = (-1)^{\#\text{swaps}}\big(\text{product of diagonal entries in REF}\big).$$

4. The determinant can be computed using any cofactor expansion.

5. $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = \det(A)^{-1}$.

6. $\det(A) = \det(A^T)$.

7. $|\det(A)|$ is the volume of the parallelepiped defined by the columns of A.

8. If A is an $n \times n$ matrix with transformation $T(x) = Ax$, and S is a subset of $\mathbf{R}^n$, then the volume of $T(S)$ is $|\det(A)|$ times the volume of S. (Even for curvy shapes S.)

9. The determinant is multi-linear (optional material).

Computing the Determinant by Row Reduction

Example first

We can use the properties of the determinant and row reduction to compute the determinant of any matrix!

$$\det\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 5 & 7 & -4\end{pmatrix} = -\det\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 5 & 7 & -4\end{pmatrix} \qquad \text{(swap)}$$

$$= -\det\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 7 & -9\end{pmatrix} \qquad \text{(`shear')}$$

$$= -\det\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & -9\end{pmatrix} \qquad \text{(`shear')}$$

$$= (-1)\cdot(-9)\det\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \qquad \text{(scale)}$$

$$= (-1)\cdot(-9)\det\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \qquad \text{(`shear')}$$

$$= 9 \qquad \text{(cube)}$$

The second matrix is obtained from the first matrix by scaling by $-1/9$. So the determinant of the first matrix is -9 times the determinant of the second matrix.

Computing the Determinant by Row Reduction

Saving some work We can stop row reducing when we get to row echelon form.

$$\det\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{pmatrix} = \cdots = -\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -9 \end{pmatrix} = 9.$$

Row reduction

This is almost always the easiest way to compute the determinant of a large, complicated matrix, either by hand or by computer.

Cofactor expansion is $O(n!) \sim O(n^n\sqrt{n})$, row reduction is $O(n^3)$.

Extra: Multi-Linearity of the Determinant

Think of det as a function of the *columns* of an $n \times n$ matrix:

$$\det\colon \underbrace{\mathbf{R}^n \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n}_{n \text{ times}} \longrightarrow \mathbf{R}$$

$$\det(v_1, v_2, \ldots, v_n) = \det \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

Multi-linear: For *any* i and any vectors $v_1, v_2, \ldots, v_n$ and v_i' *and any scalar* c,

$$\det(v_1, \ldots, v_i + v_i', \ldots, v_n) = \det(v_1, \ldots, v_i, \ldots, v_n) + \det(v_1, \ldots, v_i', \ldots, v_n)$$
$$\det(v_1, \ldots, cv_i, \ldots, v_n) = c \det(v_1, \ldots, v_i, \ldots, v_n).$$

In words: if column i is a sum of two vectors v_i, v_i', then the determinant is the sum of two determinants, one with v_i in column i, and one with v_i' in column i.

Proof: just expand cofactors along column i.

- We already knew: Scaling *one column* by c scales det by c.
- *Same properties* hold if we replace column *by row*.
- *This only works one column (or row) at a time.*

Extra: The Determinant is a Function

We can think of the *determinant as a function* of the entries of a matrix:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ \quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \end{array}$$

The formula for the determinant of an $n \times n$ matrix has $n!$ terms.

When mathematicians encounter a function whose *formula is too difficult* to write down, we try to **characterize it in terms of its properties**.

P characterizes object X

Not only does object X have property P,
but **X is the only one** thing that has property P.

Other example:

- e^x is unique function that has $f'(x) = f(x)$ and $f(0) = 1$.

Extra: Why is Property 5 true?

In Lay, there's a proof using elementary matrices. Here's another one.

Let B be an $n \times n$ matrix. There are two cases:

1. If $\det(B) = 0$, then B is not inverible. So for any matrix A, BA is not invertible. (Otherwise $B^{-1} = A(BA)^{-1}$.) So

$$\det(BA) = 0 = 0 \cdot \det(A) = \det(B)\det(A).$$

2. If A is invertible, define another function

$$f : \{n \times n \text{ matrices}\} \longrightarrow \mathbf{R} \quad \text{by} \quad f(B) = \frac{\det(BA)}{\det(A)}.$$

   Let's check the defining properties:

   1. $f(I_n) = \det(I_n A)/\det(A) = 1$.

   2–4. Doing a row operation on B and then multiplying by A, does the *same row operation* on BA. This is because a row operation is left-multiplication by an elementary matrix E, and $(EB)A = E(AB)$. Hence f scales like det with respect to row operations.

   By uniqueness, $f = \det$, i.e.,

$$\det(B) = f(B) = \frac{\det(AB)}{\det(A)} \quad \text{so} \quad \det(A)\det(B) = \det(AB).$$