Section 2.8

Subspaces of $\mathbf{R}^n$

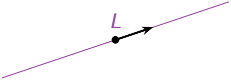
Subspaces: Motivation and examples

Example

The subset $\{0\}$: this subspace contains only one vector.

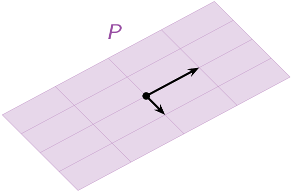
Example

A line L *through the origin*:
this contains the span of any vector in L.



Example

A plane P *through the origin*:
this contains the span of any two vectors in P.



Example

All of $\mathbf{R}^n$:
this contains 0, and is closed under addition and scalar multiplication.

- The **span was our first example** of subspace: $Span\{v_1, \ldots, v_p\}$.
- But in general, subspaces are not defined by *'the generating vectors'*

The Definition of Subspace

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. *The zero* vector *is in* V. “not empty”
2. If u *and* v are in V, *then* $u + v$ is also in V. “closed under addition”
3. If u is in V and c is in $\mathbf{R}$, *then* cu is in V. “closed under × scalars”

Consequences of definition:

- By (3), *if* v is in V, then *so is the line through* v.
- By (2),(3), *if* u, v are in V, then *so is* $xu + yv$, for all $x, y \in \mathbf{R}$.

A subspace V *contains the span* of any set *of vectors in* V.

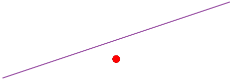
Non-Examples

Color code

Purple: wanna-be ‘subspaces’
Red vectors: **would have to be in** the subset too.

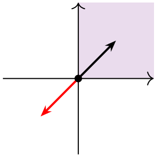
Non-Example

Any set that *doesn’t contain the origin*
Fails condition (1).



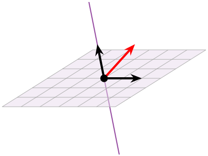
Non-Example

The first quadrant in $\mathbf{R}^2$.
Fails close under × scalar only.



Non-Example

A line union a plane in $\mathbf{R}^3$.
Fails close under addition only.



Poll

Poll

Why is the first quadrant of $\mathbf{R}^2$ not a subspace? Which property(ies) does it fail?

$$Q = \left\{ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} : a_1 \geq 0, a_2 \geq 0 \right\}$$

1. The *zero vector is contained* in the first quadrant: ✓
2. It is *closed under addition*: $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in Q$ then

$$a_1 + b_1 \geq 0 \quad a_2 + b_2 \geq 0 \Rightarrow \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix} \in Q$$ ✓

3. It is **not closed under** × scalar: let $a_1, a_2 > 0$ and $c < 0$ then

$$c \cdot a_1 < 0 \quad c \cdot a_2 < 0 \Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in Q, \quad c \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \notin Q$$ ✗

Basis and dimension of a Subspace

!!!

Every span is a subspace but also every subspace is a span.

How would you *find the generating vectors*?

Definition

Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in V such that:

1. $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is *linearly independent*.

The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Important

A subspace has many *different bases*, but they all have the *same number* of vectors (see the exercises in §2.9).

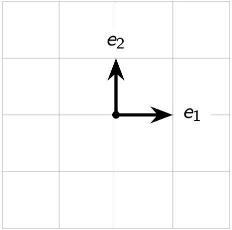
Bases of $\mathbf{R}^2$

Question

What is a basis for $\mathbf{R}^2$?

We need two vectors that *span* $\mathbf{R}^2$ and are *linearly independent*. $\{e_1, e_2\}$ is one basis.

1. They span: $\binom{a}{b} = ae_1 + be_2$.
2. They are linearly independent.

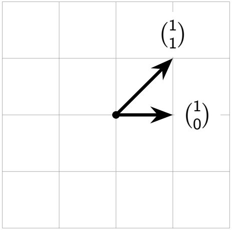


Question

What is another basis for $\mathbf{R}^2$?

Any two *nonzero vectors* that are *not collinear*. $\left\{\binom{1}{0}, \binom{1}{1}\right\}$ is also a basis.

1. They span: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has a pivot in every row.
2. They are linearly independent: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has a pivot in every column.



Bases of $\mathbf{R}^n$

The unit coordinate vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

are a basis for $\mathbf{R}^n$.

The identity matrix has columns $e_1, e_2, \ldots, e_n$.

1. They span: I_n has a pivot in every row.
2. They are linearly independent: I_n has a pivot in every column.

In general:

$\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^n$ if and only if the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

has a pivot in every row and every column, i.e. if *A is invertible*.

Example

Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x + 3y + z = 0 \right\} \qquad \mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}.$$

Verify that $\mathcal{B}$ *is a basis* for V.

0. In V: both vectors satisfy the equation, so are in V

$$-3 + 3(1) + 0 = 0 \quad \text{and} \quad 0 + 3(1) + (-3) = 0.$$

1. Span: If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in V, then $y = -\frac{1}{3}(x+z)$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{x}{3}\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} - \frac{z}{3}\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}.$$

2. Linearly independent:

$$c_1 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = 0 \implies \begin{pmatrix} -3c_1 \\ c_1 + c_2 \\ -3c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies c_1 = c_2 = 0.$$

Subspaces of a transformation

An $m \times n$ matrix A naturally gives rise to *two* subspaces.

Definition

> The *column space* of A is the subspace of $\mathbf{R}^m$ spanned by the columns of A. It is *written* $\operatorname{Col} A$.
>
> The **null space** of A is a subspace of $\mathbf{R}^n$ containing the set of all solutions of the homogeneous equation $Ax = 0$:
>
> $$\operatorname{Nul} A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$

Note: The column space is the range of the transformation $T(x) = Ax$.

Basis Nul A

The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for $\operatorname{Nul} A$.

Basis Col A

The *pivot columns* of A always form a basis for $\operatorname{Col} A$.

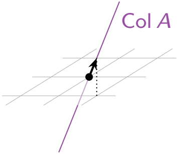
Column Space and Null Space

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Let's compute the *column space*:

$$\mathrm{Col}\,A = \mathrm{Span}\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\} = \mathrm{Span}\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}.$$



This *is a line in* $\mathbf{R}^3$.

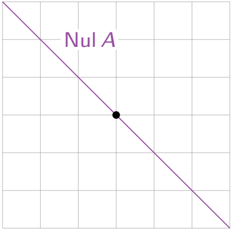
Let's compute the **null space**:

$$A\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\x+y\\x+y\end{pmatrix}.$$

This zero if and only if $x = -y$. So

$$\mathrm{Nul}\,A = \left\{ \begin{pmatrix}x\\y\end{pmatrix} \text{ in } \mathbf{R}^2 \mid y = -x \right\}.$$

This defines **a line in $\mathbf{R}^2$**:



Section 2.9

Dimension and Rank

The Rank Theorem

Definition

The **rank** of a matrix A, *written* rank A, is the *dimension of the range* of $T(x) = Ax$ (dimension of Col A).

Observe:

$$\begin{aligned} \operatorname{rank} A = \dim \operatorname{Col} A &= \text{the number of columns with pivots} \\ \dim \operatorname{Nul} A &= \text{the number of free variables} \\ &= \text{the number of columns without pivots.} \end{aligned}$$

Rank Theorem

If A is an $m \times n$ matrix, then

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n = \text{the number of columns of } A.$$

Basis Theorem

Let V be a **subspace of dimension m**. Then:

- Any m linearly independent vectors in V form *a basis* for V.
- Any m vectors that span V form *a basis* for V.

The Rank Theorem

Example

$$A = \left(\underbrace{\boxed{\begin{matrix} 1 \\ -2 \\ 2 \end{matrix}} \; \boxed{\begin{matrix} 2 \\ -3 \\ 4 \end{matrix}}}_{\text{basis of Col } A} \quad \begin{matrix} 0 & -1 \\ 4 & 5 \\ 0 & -2 \end{matrix}\right) \xrightarrow{\text{rref}} \left(\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix} \quad \underbrace{\boxed{\begin{matrix} -8 \\ 4 \\ 0 \end{matrix}} \; \boxed{\begin{matrix} -7 \\ 3 \\ 0 \end{matrix}}}_{\text{free variables}}\right)$$

A basis for Col A is

$$\left\{\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}\right\},$$

so rank $A = \dim \operatorname{Col} A = 2$.

Since there are two free variables x_3, x_4, the parametric vector form for the solutions to $Ax = 0$ is

$$x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis for Nul } A} \left\{\begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Thus $\dim \operatorname{Nul} A = 2$.

The *Rank Theorem* says $2 + 2 = 4$.

Bases as Coordinate Systems

Summary

If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ is a basis for a subspace V and x is in V, then

$$\boxed{[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad \text{means} \quad x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m.}$$

Finding the $\mathcal{B}$-coordinates for x means solving the vector equation

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

in the unknowns $c_1, c_2, \ldots, c_m$. This (usually) means row reducing the augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_m & x \\ | & | & & | & | \end{array}\right).$$

Question: What happens if you try to find the $\mathcal{B}$-coordinates of *x not in V*? You end up with an *inconsistent system*: $x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$ has no solution.

Bases as Coordinate Systems

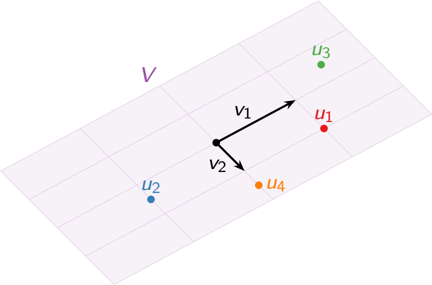
Picture

Let

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

These form *a basis $\mathcal{B}$* for the plane

$$V = \text{Span}\{v_1, v_2\}\text{in } \mathbf{R}^4.$$



Question: Estimate the *$\mathcal{B}$-coordinates* of these vectors:

$$[u_1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad [u_2]_{\mathcal{B}} = \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix} \qquad [u_3]_{\mathcal{B}} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} \qquad [u_4]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix}$$

Remark

Make sense of V as two-dim: Choose a basis $\mathcal{B}$ and use $\mathcal{B}$-coordinates.
Careful: The coordinates give *only the coefficients* of a linear combination *using such basis vectors*.

Bases as Coordinate Systems

Example

Let $v_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}$, $\quad V = \mathrm{Span}\{v_1, v_2, v_3\}$.

Question: Find a basis for V.
V is the column span of the matrix

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 3 & 1 & 8 \\ 2 & 1 & 6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis for the column span is formed by the pivot columns: $\mathcal{B} = \{v_1, v_2\}$.

Question: Find the $\mathcal{B}$-coordinates of $x = \begin{pmatrix} 4 \\ 11 \\ 8 \end{pmatrix}$.

We have to solve $x = c_1 v_1 + c_2 v_2$.

$$\left(\begin{array}{cc|c} 2 & -1 & 4 \\ 3 & 1 & 11 \\ 2 & 1 & 8 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

So $x = 3v_1 + 2v_2$ and $[x]_{\mathcal{B}} = \binom{3}{2}$.

The Invertible Matrix Theorem

Addenda

Using the *Rank Theorem* and the *Basis Theorem*, we have new interpretations of the **meaning of invertibility**.

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix, and let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. *The following statements are equivalent*.

1. **A is invertible**.
2. T is invertible.
3. A is row equivalent to I_n.
4. A has n pivots.
5. $Ax = 0$ has only the trivial solution.
6. The columns of A are linearly independent.
7. T is one-to-one.
8. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
9. The columns of A span $\mathbf{R}^n$.
10. T is onto.
11. A has a left inverse (there exists B such that $BA = I_n$).
12. A has a right inverse (there exists B such that $AB = I_n$).
13. A^T is invertible.
14. The columns of A form a basis for $\mathbf{R}^n$.
15. $\operatorname{Col} A = \mathbf{R}^n$.
16. $\dim \operatorname{Col} A = n$.
17. $\operatorname{rank} A = n$.
18. $\operatorname{Nul} A = \{0\}$.
19. $\dim \operatorname{Nul} A = 0$.

Extra: Why coefficients are unique

Lemma ← like a theorem, but less important

If $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ **is a basis** for a subspace V, then any vector x in V can be written as a linear combination

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

for **unique coefficients** $c_1, c_2, \ldots, c_m$.

Proof. We know x is a linear combination of the v_i (they span V). *Suppose that* we *can write x* as a linear combination *with different lists of coefficients*:

$$\begin{aligned} x &= c_1 v_1 + c_2 v_2 + \cdots + c_m v_m \\ x &= c_1' v_1 + c_2' v_2 + \cdots + c_m' v_m \end{aligned}$$

Subtracting:

$$0 = x - x = (c_1 - c_1')v_1 + (c_2 - c_2')v_2 + \cdots + (c_m - c_m')v_m$$

Since $v_1, v_2, \ldots, v_m$ *are linearly independent*, they only have the trivial linear dependence relation. That *means each* $c_i - c_i' = 0$, or $c_i = c_i'$.

How do you check if a subset is a subspace?

- Is it a span?
- Is it all of $\mathbf{R}^n$ or the zero subspace $\{0\}$?

Can it be written as

- a span?
- the column space of a matrix?
- the null space of a matrix?
- a type of subspace that we'll learn about later (eigenspaces, . . .)?

If so, then it's automatically a subspace.

If all else fails:

- Can you *verify directly* that it satisfies the *three defining properties*?