Why do we care about abstract definitions?

Example

The *set of polynomials of degree at most 2* is:

- $V = \{ax^2 + bx + c : a, b, c \in \mathbf{R}\}$
- V is a vector space of dimension 3
- A basis for V is $\mathcal{B} = \{v_1, v_2, v_3\}$, with $v_1 = x^2, v_2 = x, v_3 = 1$

Recall: elements in a vector space do not multiply each other.

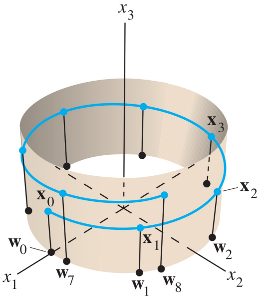
Define $T : V \to V$ by $T(ax^2 + bx + c) = 2ax + b$.

- This is the differential *operator* and *it is linear*!
- Onto? No. There is no polynomial in V with derivative x^2.
- One-to-one? No. Both $ax^2 + bx$ and $ax^2 + bx + 1$ have same derivative.

Section 5.1

Eigenvectors and Eigenvalues

Motivation: Dynamical systems



$$A = \begin{pmatrix} .8 & -.6 & 0 \\ .6 & .8 & 0 \\ 0 & 0 & 1.07 \end{pmatrix}$$

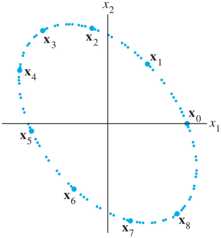
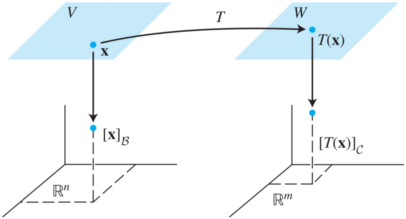


FIGURE 1 Iterates of a point $\mathbf{x}_0$ under the action of a matrix with a complex eigenvalue.

$$A = \begin{pmatrix} .5 & -.6 \\ .75 & 1.1 \end{pmatrix}$$

Recall: Rotations correspond to $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$

Background: Change of basis



$$\begin{tikzcd}
\mathbf{x} \arrow[r, "T"] \arrow[d] & T(\mathbf{x}) \arrow[d] \\
{[\mathbf{x}]_{\mathcal{B}}} \arrow[r, "\text{Multiplication}", "\text{by } M"'] & {[T(\mathbf{x})]_{\mathcal{C}}}
\end{tikzcd}$$

$\mathcal{B}$-matrix for T

When $V = W = \mathbf{R}^n$ and $\mathcal{B} = \mathcal{C}$ (vectors in the same order), then the matrix M is called the $\mathcal{B}$-matrix for T denoted by $[T]_{\mathcal{B}}$.

Background: A matrix for a given basis

Let $V = \{ax^2 + bx + c : a, b, c \in \mathbf{R}\}$ and $T(ax^2 + bx + c) = 2ax + b$.

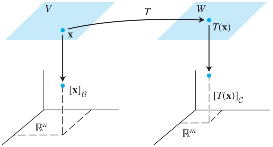
- A basis of V is $\mathcal{B} = \{x^2, x, 1\}$.

Let $M = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, if $v = ax^2 + bx + c$ then

- $[v]_{\mathcal{B}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$
- $[T(v)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 2a \\ b \end{pmatrix} = M \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

M is called the **$\mathcal{B}$-matrix for T** and denoted by $[T]_{\mathcal{B}}$.

Background: Change of basis



$$\begin{tikzcd} \mathbf{x} \arrow[r, "T"] \arrow[d] & T(\mathbf{x}) \arrow[d] \\ {[\mathbf{x}]_{\mathcal{B}}} \arrow[r, "\text{Multiplication}", "\text{by } M"'] & {[T(\mathbf{x})]_{\mathcal{C}}} \end{tikzcd}$$

If $\mathcal{B} = \{b_1, \ldots, b_n\}$ then

$$M = \big([T(b_1)]_{\mathcal{C}} [T(b_2)]_{\mathcal{C}} \ldots [T(b_n)]_{\mathcal{C}} \big)$$

Compare: Use this formula for the case $T : \mathbf{R}^n \to \mathbf{R}^n$ and $\mathcal{B} = \mathcal{C} = \{e_1, \ldots, e_n\}$.

Change of basis

Suppose $A = PDP^{-1}$. If $\mathcal{B}$ is the basis for $\mathbf{R}^n$ formed from the columns of P, then D is the **$\mathcal{B}$-matrix** for the transformation $T(x) = Ax$.

Motivation 2: Difference equations

A Biology Question

How to predict a population of rabbits with given **dynamics**:

1. half of the newborn rabbits *survive* their first year;
2. of those, half *survive* their second year;
3. their maximum *life span* is three years;
4. Each rabbit gets $0, 6, 8$ *baby rabbits* in their three years, respectively.

Approach: Each year, count the population **by age**:

$$v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} \text{ where } \begin{cases} f_n & = \text{first-year rabbits in year } n \\ s_n & = \text{second-year rabbits in year } n \\ t_n & = \text{third-year rabbits in year } n \end{cases}$$

The *dynamics say*:

$$\overbrace{\begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}}^{v_{n+1}} = \begin{pmatrix} 6s_n + 8t_n \\ f_n/2 \\ s_n/2 \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}}^{Av_n}.$$

Motivation: Difference equations
Continued

This is a **difference equation**: $Av_n = v_{n+1}$

If you know *initial population* v_0, what happens *in 10 years* v_{10}?

Plug in a computer:

v_0	v_{10}	v_{11}
$\begin{pmatrix}1\\2\\3\end{pmatrix}$	$\begin{pmatrix}9459\\2434\\577\end{pmatrix}$	$\begin{pmatrix}19222\\4729\\1217\end{pmatrix}$
$\begin{pmatrix}3\\7\\9\end{pmatrix}$	$\begin{pmatrix}30189\\7761\\1844\end{pmatrix}$	$\begin{pmatrix}61316\\15095\\3881\end{pmatrix}$
$\begin{pmatrix}16\\4\\1\end{pmatrix}$	$\begin{pmatrix}16384\\4096\\1024\end{pmatrix}$	$\begin{pmatrix}32768\\8192\\2048\end{pmatrix}$

Notice any patterns?

1. Each segment of the population *essentially doubles* every year: $Av_{11} \approx 2v_{10}$.
2. The ratios get close to $(16 : 4 : 1)$:

$$v_{11} \approx (\text{big}\#) \cdot \begin{pmatrix}16\\4\\1\end{pmatrix}.$$

New terms coming: eigenvalue, and eigenvector

Motivation: Difference equations

Continued (2)

If v_0 satisfies $A_{v_0} = \lambda v_0$ then

$$v_n = A^{n-1}(Av_0) = \lambda A^{n-1} v_0 = \lambda^2 A^{n-2} v_0 \qquad \ldots = \lambda^n v_0.$$

It is **much easier** to compute $v_n = \lambda^{10} v_0$.

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad v_0 = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} \qquad Av_0 = 2v_0.$$

Starting with 16 baby rabbits, 4 first-year rabbits, and 1 second-year rabbit:

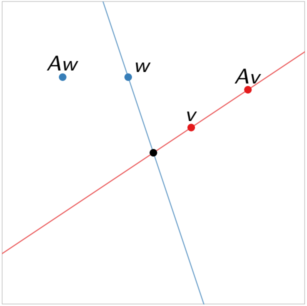
- The population will exactly double every year,
- In 10 years, you will have $2^{10} \cdot 16$ baby rabbits, $2^{10} \cdot 4$ first-year rabbits, and 2^{10} second-year rabbits.

Geometrically

Eigenvectors

An eigenvector of a matrix A is *a nonzero vector v* such that:

- Av is a multiple of v, which means
- *Av is on the same line* as v.



v is an eigenvector

w is not an eigenvector

Poll

Poll

Which of the vectors

A. $\begin{pmatrix}1\\1\end{pmatrix}$ B. $\begin{pmatrix}1\\-1\end{pmatrix}$ C. $\begin{pmatrix}-1\\1\end{pmatrix}$ D. $\begin{pmatrix}2\\1\end{pmatrix}$ E. $\begin{pmatrix}0\\0\end{pmatrix}$

are eigenvectors of the matrix $\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$?

$\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix}$ eigenvector with eigenvalue 2

$\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = 0\begin{pmatrix}1\\-1\end{pmatrix}$ eigenvector with eigenvalue 0

$\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = 0\begin{pmatrix}-1\\1\end{pmatrix}$ eigenvector with eigenvalue 0

$\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}$ not an eigenvector

$\begin{pmatrix}0\\0\end{pmatrix}$ is never an eigenvector

Eigenvectors and Eigenvalues

Definitions

If v **is not zero** and $Av = \lambda v$ then
v is *an eigenvector* and λ is *its eigenvalue*.

- Eigenvalues and eigenvectors are only for square matrices.
- Eigenvectors are by definition nonzero. *Eigenvalues may be equal to zero.*

Example

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4v$$

Hence v is an *eigenvector* of A, with *eigenvalue* $\lambda = 4$.

Verifying Eigenvalues

Question: Is $\lambda = 3$ an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$?

In other words, does

$$\left.\begin{aligned} &Av = 3v \\ &Av - 3v = 0 \\ &(A - 3I)v = 0 \end{aligned}\right\} \text{have a nontrivial solution?}$$

We know how to answer that: Row reduction!

$$A - 3I = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

Parametric vector form: $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} -4 \\ 1 \end{pmatrix}$.

Bottom-line: Any nonzero multiple of $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ is an *eigenvector with eigenvalue* $\lambda = 3$ Check one of them.

Eigenspaces

The **λ-eigenspace** is a *subspace* of $\mathbf{R}^n$ containing all *eigenvectors of A with eigenvalue λ, plus the zero* vector:

$$\lambda\text{-eigenspace} = \mathrm{Nul}\big(A - \lambda I\big).$$

Find a basis for the 2-eigenspace of

λ

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

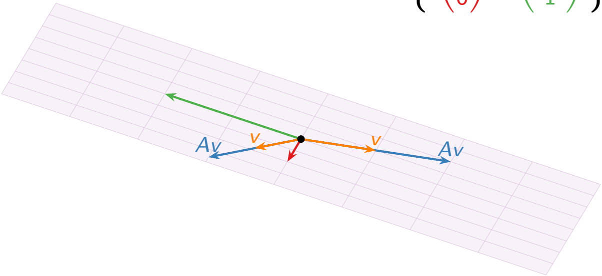
$$A - 2I = \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{parametric vector form}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_2 \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + v_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

$$\xrightarrow{\text{basis}} \left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

This is how eigenvalues and eigenvectors make matrices easier to understand.

What does this 2-eigenspace look like? A basis is $\left\{ \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.



For any v in the 2-eigenspace, $Av = 2v$ by definition.
This means, on its 2-eigenspace, A acts by *scaling by* 2.

Summary

Let A be an $n \times n$ matrix and let λ be a number.

1. λ is an **eigenvalue of A**
   if and only if $(A - \lambda I)x = 0$ *has a nontrivial solution*.

2. Finding a basis for the **λ-eigenspace of A**
   means finding a basis for $\mathrm{Nul}(A - \lambda I)$,

3. The **eigenvectors** with eigenvalue λ are
   the nonzero elements of $\mathrm{Nul}(A - \lambda I)$

- If we *know λ is eigenvalue*: easy to find eigenvectors (row reduction).
- And to **find all eigenvalues**? Will need to *compute a determinant*.
  Finding λ that has a non-trivial solution to $(A - \lambda I)v = 0$ boils down to finding λ that makes $\det(A - \lambda I) = 0$.

Some facts you can work out yourself

Fact 1

A is **invertible** if and only if 0 *is not an eigenvalue* of A.

Fact 2

If $v_1, v_2, \ldots, v_k$ are eigenvectors of A with **distinct eigenvalues** $\lambda_1, \ldots, \lambda_k$, then $\{v_1, v_2, \ldots, v_k\}$ is *linearly independent*.

Consequence of Fact 2

An $n \times n$ matrix has **at most** n distinct eigenvalues.

Fact 3

The **eigenvalues** of a triangular matrix are the *diagonal entries*.

Extra: Some proofs

Why Fact 1?

$$\begin{aligned} 0 \text{ is an eigenvalue of } A &\iff Ax = 0 \text{ has a nontrivial solution} \\ &\iff A \text{ is not invertible.} \end{aligned}$$

Why Fact 2 (for two vectors)?

If v_2 is a multiple of v_1, then v_2 is contained in the λ_1-eigenspace. This is not true as v_2 does not have the same eigenvalue as v_1.

Example

Find all eigenvalues of $A = \begin{pmatrix} 3 & 4 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{pmatrix}$.

$$A - \lambda I = \begin{pmatrix} 3 & 4 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{pmatrix} - \lambda I_3 = \begin{pmatrix} 3-\lambda & 4 & 1 \\ 0 & -1-\lambda & -2 \\ 0 & 0 & 3-\lambda \end{pmatrix}$$

Since $\det(A - \lambda I) = (3-\lambda)^2(-1-\lambda)$, eigenvalues are $\lambda = 3$ and -1.