**Math 2802 N1-N3 Worksheet 11**

Solutions

**1.** Circle **T** if the statement is always true and circle **F** if it is ever false. The matrices here are $n \times n$.

a) **T** **F** If $A$ is symmetric and has an eigenvalue $\lambda$, then there is a unit-length vector $x$ such that $\lambda \leq x^T Ax$.

b) **T** **F** If $A$ is an $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then the condition number equals $\lambda_1/\lambda_n$.

## Solution.

**a) True.** Let $Q(x) = x^T Ax$, then the maximum value of $Q(x)$ constrained to vectors of length one ($x^T x = 1$) is attained by the unit-length eigenvector of the largest eigenvalue of $A$. Call this largest eigenvalue $\lambda_1$ and such eigenvector $v_1$. Then

$$v_1^T Av_1 = v_1^T(\lambda_1 v_1) = \lambda_1(v_1^T v_1) = \lambda_1 \geq \lambda$$

**b) False.** The condition number is defined in terms of the singular values; these are ordered from largest to smallest $\sigma_1 \geq \cdots \sigma_n$ and the condition number equals $\sigma_1/\sigma_n$. However, the singular values correspond to the **square roots** of the **eigenvalues of** $A^T A$.

**2.** For the quadratic functions below, find the vector $u$ attaining the maximum value of $Q(x) = x^T Ax$ among vector of unit length; i.e. constrained to have $x^T x = 1$.

**a)** $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$

**b)** $A = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$

## Solution.

**a)** The characteristic polynomial is given by

$$\begin{aligned} \det(A - \lambda I) &= (3-\lambda)(2-\lambda)^2 + 2 + 2 - [(2-\lambda) + 4(3-\lambda) + (2-\lambda)] \\ &= -\lambda(\lambda-5)(\lambda-2) \end{aligned}$$

An eigenvector of 5 is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Thus, the maximum value of $Q(x)$ for unit-length vectors is 5 and a unit vector attaining this value is $u = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

**b)** The characteristic polynomial is given by

$$\det(A-\lambda I) = (3-\lambda)^2 - 4 = (\lambda-5)(\lambda-1)$$

An eigenvector of 5 is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Thus, the maximum value of $Q(x)$ for unit-length vectors is 5 and a unit vector attaining this value is $u = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

**3.** Find the singular value decomposition $A = U\Sigma V^T$ of $A = \begin{pmatrix} 7 & 6 \\ 0 & 0 \\ 6 & 2 \end{pmatrix}$

## Solution.

**Step zero:** Compute $A^TA$, its eigenvalues and eigenvectors (listed in decreasing order of eigenvalues). Since the eigenvalues have all multiplicity one; we already have an orthogonal diagonalization of $B = A^TA = PDP^T$.

$$A^TA = \begin{pmatrix} 85 & 54 \\ 54 & 40 \end{pmatrix} = \begin{pmatrix} 3\sqrt{13} & -2\sqrt{13} \\ 2\sqrt{13} & 3\sqrt{13} \end{pmatrix}\begin{pmatrix} 121 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 3/\sqrt{13} & 2/\sqrt{13} \\ -2\sqrt{13} & 3\sqrt{13} \end{pmatrix}$$

The singular values of $A$ are, thus, $\sigma_1 = \sqrt{121} = 11$ and $\sigma_2 = \sqrt{4} = 2$.

**Step one:** Let $V = P$; and let the column vectors of $V$ be $v_1, v_2$.

$$v_1 = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad v_2 = \frac{1}{\sqrt{13}}\begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

**Step two:** Let $U = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}$ as follows. Use the non-zero singular values to define vectors in $\mathbf{R}^3$:

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{1}{11\sqrt{13}}\begin{pmatrix} 33 \\ 0 \\ 22 \end{pmatrix} = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$$

$$u_2 = \frac{Av_2}{\sigma_2} = \frac{1}{2\sqrt{13}}\begin{pmatrix} 4 \\ 0 \\ -6 \end{pmatrix} = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

And complete the basis with a unit vector

$$u_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

If this selection of $u_3$ is not straigth-forward, then Gram-Schmidt might be useful.

(Note that for singular values that are zero, the definition $\frac{Av_1}{\sigma_1}$ has a division by zero, so it would not be defined.)

**Step four:** Let $\Sigma$ be a matrix with same number of rows and columns of $A$ (this will not always be a diagonal matrix); fill in the *'diagonal'* entries with the non-zero singular values and leave the rest of the entries equal zero.

$$\Sigma = \begin{pmatrix} 11 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}.$$