MATH 2802 N1-N3, WORKSHEET 2

JANUARY 19TH, 2018

(1) **True or False** (Justify your answer)
If a system of equations has more variables than equations then it must be consistent.

Solution. In this case, the corresponding matrix will have more columns (variables) that rows (equations). However, the matrix could still row reduce to a matrix with a row of zeros.
Eg. $A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(2) Let $v_1 = \begin{bmatrix} 12 \\ 1 \\ 2 \\ 6 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ -1 \\ 1 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 10 \\ 1 \\ 5 \\ 10 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $v_5 = \begin{bmatrix} -10 \\ -1 \\ -5 \\ -10 \end{bmatrix}$

(a) What is the shape of $Span\{v_1, v_2, v_3, v_4\}$?
(b) What is the shape of $Span\{v_1, v_2, v_3, v_5\}$?
(c) Is it possible to find vectors $w_1, w_2, \dots, w_p$ in $\mathbb{R}^{p+1}$ that span all $\mathbb{R}^{p+1}$? (Justify your answer)

Solution.
(a),(b) Both $Span\{v_1, v_2, v_3, v_4\}$ and $Span\{v_1, v_2, v_3, v_5\}$ are 3D subspaces and are equal to $Span\{v_1, v_2, v_3\}$
Why? We can inspect the vectors one by one: v_1 is not the zero vector, so it spans a line; v_2 is not collinear to v_1 so together, v_1 and v_2 spans a plane. Next we have to row reduce matrix $A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ to see if v_1, v_2, v_3 are linearly dependent (alternatively, row reduce $A = \left(\begin{array}{cc|c} v_1 & v_2 & v_3 \end{array}\right)$ to find out if v_3 can be written as a linear combination of v_1 and v_2). The test consists of looking for non-pivot columns.

For the row reduction:

$$\begin{pmatrix} 12 & 6 & 10 \\ 1 & -1 & 1 \\ 2 & 1 & 5 \\ 6 & 2 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 12 & 6 & 10 \\ 2 & 1 & 5 \\ 6 & 2 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 18 & -2 \\ 0 & 3 & 3 \\ 0 & 8 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 18 & -2 \\ 0 & 8 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -14 \\ 0 & 0 & -4 \end{pmatrix}$$

Since all columns have a pivot, we conclude that v_1, v_2, v_3 are linearly independent and thus $Span\{v_1, v_2, v_3\}$ (alternatively, v_3 is not a linear combination of v_1, v_2, so the span gets bigger, from plane to 3D).
Now, v_4 is the zero vector, so v_4 is already part of $Span\{v_1, v_2, v_3\}$; similarly, v_5 is colinear to v_3 so it is already part of $Span\{v_1, v_2, v_3\}$.

(c) Place vectors $v_1, \dots, v_p$ in a $(p+1) \times p$ matrix A. These vectors span $\mathbf{R}^{p+1}$ if $Ax = b$ is consistent for all b in $\mathbf{R}^{p+1}$ (that is, all vectors in $\mathbf{R}^{p+1}$ are linear combinations of $v_1, \dots, v_p$.
Since there are only p columns, there are at most p pivots in the echelon form of A. Therefore, there will be at most one row with no pivots, eg. one row with all zeros. We can find a b for wich $Ax = b$ is inconsistent.

(3) Consider the following augmented matrices corresponding systems of linear equations (line separating last column missing). Find which ones are consistent, which ones have exactly one solutions; if the system has more than one solution then write down the solution set in parametric vector form.

$$\left(\begin{array}{ccc|c} 1 & 4 & -4 & 3 \\ 0 & 2 & 5 & 4 \\ 0 & 0 & -3 & 5] \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 7 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 7 & 0 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solution.

(a) The first matrix has three pivots thus, it is consist and it has a unique solution.
The other two matrices are consistent (because the last entry of the augmented side of the matrix is also a zero entry). Since there are non-pivot columns, then there is more than one solution.

(b) Second matrix has x_2 as free variable and the row reduction translates to

$$\begin{aligned} x_1 + 7x_2 &= 1 \\ x_2 &= x_2 \\ 2x_3 &= -1 \end{aligned}$$

The solution in parametric vector form is the expression on the right-hand side of

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1-7x_2 \\ x_2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1/2 \end{pmatrix} + x_2 \begin{pmatrix} -7 \\ 1 \\ 0 \end{pmatrix}$$

(c) Third matrix has x_3 as free variable and the row reduction translates to

$$\begin{aligned} x_1 + 7x_3 &= 1 \\ x_2 - 2x_3 &= -1 \\ x_3 &= x_3 \end{aligned}$$

The solution in parametric vector form is the expression on the right-hand side of

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1-7x_3 \\ -1+2x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix}$$

(4) Zander has challenged you to find his hidden treasure, located at some point (a, b, c). He has honestly guaranteed you that the treasure can be found by starting at the origin and taking steps in directions given by

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$$

By decoding Zander's message, you have discovered that the treasure's first and second entries are (in order) -4 and 3.

(a) What is the treasure's full location?
(b) Give instructions for how to find the treasure only moving in the directions given by v_1, v_2, v_3.

Solution

(a) The location of the treasure is $\begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix}$:

We translate this problem into linear algebra. Let c be the final entry of the treasure. Since Zander has assured us that we can find the treasure using the three vectors we have been given, our problem is to find c so that $\begin{pmatrix} -4 \\ 3 \\ c \end{pmatrix}$ is a linear combination of v_1, v_2 and v_3 (in other words, find c so that the treasure's location is in $Span\{v_1, v_2, v_3\}$. We form an augmented matrix and find when it gives a consistent system.

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & c-5 \end{array}\right)$$

This system will be consistent if and only if the right column is not a pivot column, so we need $c - 5 = 0$; that is $c = 5$.

(b) Getting to the point $\begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix}$ using vectors v_1, v_2, v_3 is equivalent to finding scalars x_1, x_2, x_3 such that

$$\begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

Translating to the augmented matrix formulation, we have (now we know $c = 5$)

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 7 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The general solution to this system is given in (c) of the problem above. Since the system has infinitely many solutions, there are infinitely many ways to get to the treasure. If we choose the path corresponding to $x_3 = 0$, then $x_1 = 1$ and $x_2 = -1$, which means that we move 1 unit in the direction of v_1 and -1 unit in the direction of v_2. In equations:

$$\begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + 0 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$