Math 2802 N1-N3 Worksheet 4

Solutions

1. Consider a transformation $T : \mathbf{R}^n \to \mathbf{R}^m$. Choose the correct answer and give examples in the follow up questions.

a) We guarantee that

- T is not onto if $m > n$.
  Follow up: Provide 2 examples of $T : \mathbf{R}^3 \to \mathbf{R}^2$ where *i)* T is onto *ii)* T is not onto.

- T is not onto if $m < n$.
  Follow up: Provide 2 examples of $T : \mathbf{R}^2 \to \mathbf{R}^3$ where *i)* T is onto *ii)* T is not onto.

b) We guarantee that

- T is not one-to-one if $m > n$.
  Follow up: Provide 2 examples of $T : \mathbf{R}^3 \to \mathbf{R}^2$ where *i)* T is one-to-one *ii)* T is not one-to-one.

- T is not one-to-one if $m < n$.
  Follow up: Provide 2 examples of $T : \mathbf{R}^2 \to \mathbf{R}^3$ where *i)* T is one-to-one *ii)* T is not one-to-one.

Solution.

a) We guarantee that T is not onto if $m > n$: the corresponding matrix A has more rows than columns, thus all echelon forms have rows of zeros. We can find vectors b which would make any system $Ax = b$ inconsistent.

Examples: $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is onto

$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is not onto.

b) We guarantee that T is not one-to-one if $m < n$: the corresponding matrix A has more columns than rows, thus all echelon forms have non-pivot columns. Those columns will correspond to free variables that provides, for consistent systems $Ax = b$, infinitely many solutions.

Examples: $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is one-to-one.

$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is not one-to-one.

2. Let $A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ and $A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

a) Compute the inverse of A_3 using the algorithm from class.

b) Can you guess the inverse of A_4?

c) Can you guess the inverse of the corresponding A_n?

Solution.

a) Compute the inverse of A_3 using the algorithm from class.

We place the augmented matrix $(A|I_3)$ and row reduce (line between matrices is omitted).

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Thus $A_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

b) We can guess: a diagonal of ones and a 'second diagonal' of negative ones? (or preform the same algorithm)

$$A_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

If we guessed, then we have to verify that either

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

or

$$A_4 \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

c) In general, A_n^{-1} can be described as lower triangular matrices with a diagonal of ones and a 'second diagonal' of negative and the rest of entries equal to zero. (It can be proved by induction).

3. If we know that

$$A = \begin{pmatrix} 5 & 2 & -1/2 \\ -3/2 & -7/10 & 1/5 \\ -1/2 & -1/10 & 1/10 \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} 1 & 3 & -1 \\ -1 & -5 & 5 \\ 4 & 10 & 10 \end{pmatrix}$$

Solve the system $Ax = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$.

Solution.

To solve the system we have to multiply the matrix equation with A^{-1} by the left $A^{-1}Ax = A^{-1}\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$; since $A^{-1}A = I_3$ we can simply compute

$$x = A^{-1}\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & -1 \\ -1 & -5 & 5 \\ 4 & 10 & 10 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -16 \\ -26 \end{pmatrix}$$

4. **Exercise 31 from section 2.5** For this problem, use the interactive row reducer to row reduce the matrix A (MATHLAB might use a permutated lower triangular matrix): http://people.math.gatech.edu/~jrabinoff6/1718F-1553/demos/rrinter.html

Solution.

$$A = \begin{pmatrix} 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 4 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 4 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \frac{1}{4} & 1 & & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{-1}{15} & 1 & & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-4}{15} & \frac{-8}{28} & 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & \frac{-4}{14} & \frac{-1}{12} & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & \frac{-7}{24} & \frac{-26}{89} & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \frac{-24}{89} & \frac{-4147}{48149} & 1 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \frac{-319}{1082} & \frac{-56977}{194370} & 1 & \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{15}{4} & \frac{-1}{4} & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{56}{15} & \frac{-16}{15} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{24}{7} & \frac{-2}{7} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{89}{24} & \frac{-13}{12} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1082}{319} & \frac{-26}{89} & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{930}{251} & \frac{-227}{209} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{359}{106} \end{pmatrix}$$

$$A^{-1} \sim \begin{pmatrix} 0.30 & 0.09 & 0.10 & 0.05 & 0.03 & 0.02 & 0.01 & 0.01 \\ 0.09 & 0.30 & 0.05 & 0.09 & 0.02 & 0.03 & 0.01 & 0.01 \\ 0.09 & 0.05 & 0.33 & 0.11 & 0.10 & 0.06 & 0.03 & 0.02 \\ 0.05 & 0.09 & 0.11 & 0.33 & 0.06 & 0.10 & 0.02 & 0.03 \\ 0.03 & 0.02 & 0.10 & 0.06 & 0.33 & 0.11 & 0.09 & 0.05 \\ 0.02 & 0.03 & 0.06 & 0.10 & 0.11 & 0.33 & 0.05 & 0.09 \\ 0.01 & 0.01 & 0.03 & 0.02 & 0.09 & 0.05 & 0.30 & 0.09 \\ 0.09 & 0.01 & 0.02 & 0.03 & 0.05 & 0.09 & 0.09 & 0.30 \end{pmatrix}$$