Math 2802 N1-N3 Worksheet 6

Solutions

1. Determine whether the following statements are true or give a counterexample. Let A be $n \times n$ matrix

a) If $Ax = \lambda x$ then λ is an eigenvalue of A

b) If there are matrices P, D such that $A = PDP^{-1}$ then A is diagonalizable.

c) If A has n distinct eigenvalues then A is diagonalizable.

d) If A has only one eigenvalue with algebraic multiplicity n then A is not diagonalizable.

Solution.

a) **False.** It's important to verify $x \neq 0$, otherwise we can find that any real number becomes eigenvalue of A

b) **False.** It is important to be precise about the assumption that D is a diagonal matrix.

c) **True.** If A has n distinct eigenvalues then there will be a basis of $\mathbf{R}^n$ composed of eigenvectors of A.

d) **False.** If $A = I$ is the identity matrix, then it as only one eigenvalue: 1 and it is diagonal matrix.

2. **Discuss:** What is the difference between algebraic multiplicity of an eigenvalue and geometric multiplicity of an eigenvalue.

Solution.

The algebraic multiplicity of an eigenvalue, say e.g. $\lambda = 1$, is obtained by looking at how many factors $\lambda - 1$ there are. The algebraic multiplicity is an upper bound to the dimension of the 1-eigenspace. The geometric multiplicity of eigenvalue 1 is precisely the dimension of the 1-eigenspace.

3. Find the algebraic multiplicity and eigenspace of eigenvalue 5 for matrix

$$A = \begin{pmatrix} 5 & 5 & 0 & 2 \\ 0 & 2 & -3 & 6 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Solution.

Since A is upper-triangular (and so is $A - \lambda I$) we have that $\det(A - \lambda I) = (5-\lambda)^2(2-\lambda)(3-\lambda)$. Thus, eigenvalue 5 has algebraic multiplicity 2.

The 5-eigenspace is precisely the solution set to $(A-5I)x=0$, so we have to row reduce

$$A-5I=\begin{pmatrix}0&5&0&2\\0&-3&-3&6\\0&0&-2&-2\\0&0&0&0\end{pmatrix}\sim\begin{pmatrix}0&1&1&-2\\0&0&1&1\\0&5&0&2\\0&0&0&0\end{pmatrix}$$

$$\sim\begin{pmatrix}0&1&1&-2\\0&0&1&1\\0&0&-5&12\\0&0&0&0\end{pmatrix}\sim\begin{pmatrix}0&1&1&-2\\0&0&1&1\\0&0&0&17\\0&0&0&0\end{pmatrix}$$

There is only 1 free variable, so the 5-eigenspace will have dimension 1. Solving the systems gives that all solutions have the form $\begin{pmatrix}x_1\\0\\0\\0\end{pmatrix}$. In other words, the 5-eigenspace is:

$$\left\{\begin{pmatrix}x\\0\\0\\0\end{pmatrix}: x\in\mathbf{R}\right\}=Span\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\}$$

4. Let $A=PDP^{-1}$ with $P=\begin{pmatrix}3&-1\\1&3\end{pmatrix}$ and $D=\begin{pmatrix}2&0\\0&1/2\end{pmatrix}$. Draw the eigenspaces of 2 and 1/2; and (approximately) draw x,Ax,A^2x,A^3x,A^4x; for $x=\begin{pmatrix}0\\10\end{pmatrix},\begin{pmatrix}2\\4\end{pmatrix}$. If possible, do not compute powers of A.

Solution.

Let $\mathcal{B}=\{v_1,v_2\}$ with $v_1=\begin{pmatrix}3\\1\end{pmatrix}$ and $v_2=\begin{pmatrix}-1\\3\end{pmatrix}$. Write $x_0=\begin{pmatrix}0\\10\end{pmatrix}, x_1=Ax,\ldots$, and $z_0=\begin{pmatrix}2\\4\end{pmatrix}, z_1=Az,\ldots$.

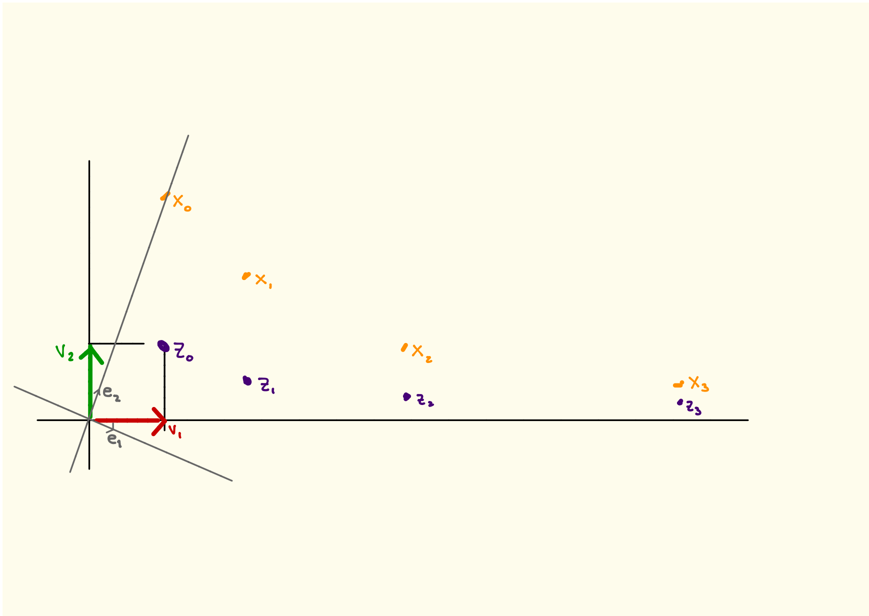
Instead of computing powers of A. Imagine that you change your basis from e_1,e_2 to v_1,v_2. Since v_1,v_2 are eigenvectors of the transformation $T(x)=Ax$, then these become the axis and the transformation only stretches/srinks along the axis (using the information of the diagonal matrix D).

In the new basis $\mathcal{B}$, the coordinates are

$$[x_0]_{\mathcal{B}}=\begin{pmatrix}1\\3\end{pmatrix},[x_1]_{\mathcal{B}}=\begin{pmatrix}2\\3/2\end{pmatrix},[x_2]_{\mathcal{B}}=\begin{pmatrix}4\\3/4\end{pmatrix},[x_3]_{\mathcal{B}}=\begin{pmatrix}8\\3/8\end{pmatrix},[x_4]_{\mathcal{B}}=\begin{pmatrix}16\\3/16\end{pmatrix}$$

$$[z_0]_{\mathcal{B}}=\begin{pmatrix}1\\1\end{pmatrix},[z_1]_{\mathcal{B}}=\begin{pmatrix}2\\1/2\end{pmatrix},[z_2]_{\mathcal{B}}=\begin{pmatrix}4\\1/4\end{pmatrix},[z_3]_{\mathcal{B}}=\begin{pmatrix}8\\1/8\end{pmatrix},[z_4]_{\mathcal{B}}=\begin{pmatrix}16\\1/16\end{pmatrix}$$

In a drawing is easier to visualize, you just have to rotate the paper:



5. Let $A = PDP^{-1}$ with $P = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Compute A^k. Can you guess what are possible dynamics for $x, Ax, A^2x, \ldots$ depending on the values of a and b?

Solution.

Note that $A^k = (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})$ using k factors. By taking out the parenthesis we notice that most of the matrices cancel: $P^{-1}P = I$. And we get $A^k = PD^kP^{-1}$. This multiplication is easy to compute.

$$A^k = PD^kP^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a^k & 2b^k \\ 0 & b^k \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^k & -2a^k + b^k \\ 0 & b^k \end{pmatrix}.$$