**Math 2802 N1-N3 Worksheet 7**

Solutions

**1.** Determine whether the following statements are true or explain why are they false (can use an example).

**a)** Every stochastic matrix has a unique steady vector.

**b)** If $Pq = q$ then for $\{x_k\}_{k\geq 0}$ (recall $x_k = Px_{k-1}$) converges to $q$ regardless of the initial point $x_0$.

## Solution.

**a)** **False.** This is only true when $P$ is a regular stochastic matrix. For example, the matrix $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}$ is not regular because all power $P^k = P$ have entries that are zeros. We can see that both $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$ are steady-state vectors.

**b)** **False.** This is only true when $P$ is a regular stochastic matrix. For example, if $q = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$ and $x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ then all vectors $x_k = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, which will not approach $q$.

**2.** Consider a Markov Chain on $\{1, 2, 3\}$ with transition matrix $P = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}$

**a)** Show that $P$ is a regular matrix

**b)** Find a steady-state vector for this markov chain

**c)** What fraction of the time does this chain spend in state 2?

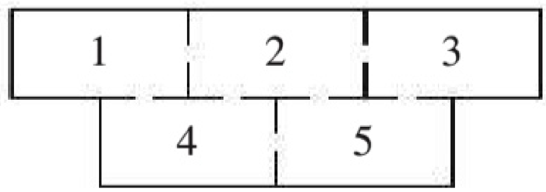
## Solution.

**a)** Computing $P^2$ we can see that there are no zero-entries. Alternatively, we can draw a 'transition graph' and see that every pair of states is at 2-step distance from each other.

**b)** Let $v = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$, then $Pv = v$

**c)** The chain will spend, on average, the same amount of time at each state: 33%

**3.** Conisder a mouse traversing from room to room (uniformly) at random in the maze below. What fraction of time does it spend in room 3.




**Solution.**

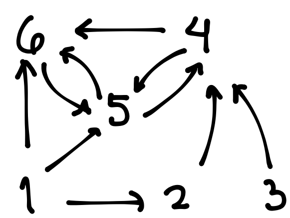
The transition matrix is $P = \begin{pmatrix} 0 & 1/4 & 0 & 1/3 & 0 \\ 1/2 & 0 & 1/2 & 1/3 & 1/3 \\ 0 & 1/4 & 0 & 0 & 1/3 \\ 1/2 & 1/4 & 0 & 0 & 1/3 \\ 0 & 1/4 & 1/2 & 1/3 & 0 \end{pmatrix}$

we approximate the steady vector by computing a large power of $P$, for example,

$$P^{100} \sim \begin{pmatrix} .15 & .15 & .15 & .15 & .15 \\ .28 & .28 & .28 & .28 & .28 \\ .15 & .15 & .15 & .15 & .15 \\ .21 & .21 & .21 & .21 & .21 \\ .21 & .21 & .21 & .21 & .21 \end{pmatrix}.$$

This also shows that $P$ is a regular matrix, and therefore, the steaday vector does represent the approximate fraction of time the mouse spends in each of the rooms. So the mouse spends about 28% of the time in room 2.

**4.** Design the google matrix for the following web.




**Solution.**

The transition matrix of a random walk in the above web is

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/2 & 0 & 1 \\ 1/3 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

But this matrix is not regular (e.g. you can see this from the zero rows), to make it regular, we add a matrix $K$ full of $1/6$ entries. This corresponds to a 'reboot': from time to time a surfer simply starts over at a random webpage.

The google matrix is: $G = .85P + .15K$ where

$$K = \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}$$

**5.** Consider the production model $x = Cx + d$ for an economy with two sectors, where $C = \begin{pmatrix} 0 & .5 \\ .6 & .2 \end{pmatrix}$ and $d = \begin{pmatrix} 50 \\ 30 \end{pmatrix}$. Determine the production level $x$ necessary to satisfy the final demand $d$.

## Solution.

Consider the matrix $(I - C) = \begin{pmatrix} 1 & -.5 \\ -.6 & .8 \end{pmatrix}$. We can verify this matrix is invertible since $det(I - C) = .5$.

Therefore, we can solve the equation $x = Cx + d$, by solving $(I - C)x = d$. In turn, we can find

$$x = (I - C)^{-1}d = \frac{1}{\det(I - C)} \begin{pmatrix} .8 & .5 \\ .6 & 1 \end{pmatrix} \begin{pmatrix} 50 \\ 30 \end{pmatrix} = \begin{pmatrix} 110 \\ 120 \end{pmatrix}$$