Math 2802 N1-N3 Worksheet 8

Solutions

1. Find an upper triangular matrix R such that $A = QR$. Check your work.

$$A = \begin{pmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{pmatrix}, \qquad Q = \frac{1}{7}\begin{pmatrix} -2 & 5 \\ 5 & 2 \\ 2 & -4 \\ 2 & 4 \end{pmatrix}$$

Solution.

Let $v_1 = \begin{pmatrix} -2 \\ 5 \\ 2 \\ 4 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3 \\ 7 \\ -2 \\ 6 \end{pmatrix}$. Then the orthogonal basis obtained from the Gram-Schmidt process is $u_1 = v_1$ and

$$u_2 = v_2 - proj_{v_1}(v_2) = v_2 - \frac{v_1 \cdot v_2}{v_1 \cdot v_1} v_1 = v_2 - \frac{49}{49} v_1 = v_2 - v_1 = \begin{pmatrix} 5 \\ 2 \\ -4 \\ 2 \end{pmatrix}$$

The columns of $\widehat{R}$ represent the equations $v_1 = u_1$ and $v_2 = u_1 + u_2$. So

$$\widehat{Q} = \begin{pmatrix} | & | \\ u_1 & u_2 \\ | & | \end{pmatrix} \text{ and } \widehat{R} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

We have to divide the first column of $\widehat{Q}$ by $||u_1||$ and multiply the first row of $\widehat{R}$ by $||u_1||$. Likewise, we have to divide the second column of $\widehat{Q}$ by $||u_2||$ and multiply the second row of $\widehat{R}$ by $||u_2||$. So we have that $R = \begin{pmatrix} 7 & 7 \\ 0 & 7 \end{pmatrix}$.

2. Apply the Gram-Schmidt decomposition to the vectors

a) $v_1 = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

b) $v_1 = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ -6 \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

Solution.

a) Start by $u_1 = v_1$, then $u_2 = v_2 - proj_{span v_1}(v_2) = v_2$ (note that v_1 and v_2 are orthogonal). Now, $u_3 = v_3 - proj_{span v_1}(v_3) - proj_{span v_2}(v_3)$, that is

$$u_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}}{\begin{pmatrix} -1 & 3 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}} \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{4}{10}\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 1/5 \\ 0 \end{pmatrix}$$

b) In this problem we find $u_1 = v_1$, $u_2 = v_2$. Now, when dealing with v_3 we will find that $u_3 = v_3 - proj_{span v_1}(v_3) - proj_{span v_2}(v_3) = 0$; this is explained beacuse v_3 is a linear combination of v_1 and v_2. For the Gram-Schimdt process, we simply discard this vector and continue with the next vector.

3. Decide whether the following sets form a basis for $\mathbf{R}^3$. If so, is it an orthogonal basis? If so, convert the orthogonal basis into an orthonormal basis.

a) $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

b) $\left\{ \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} -3 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

c) $\left\{ \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \end{pmatrix} \right\}$

Solution.

a) This is not a basis because the vectors are not linearly independent: the second vector is the sum of the order two vectors.

b) This is an orthogonal basis, its orthonormal version is:

$$\left\{ \frac{1}{\sqrt{19}}\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, \frac{1}{\sqrt{10}}\begin{pmatrix} -3 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

c) This is a basis because by row reducing $A = \begin{pmatrix} 1/3 & 1/2 & 1/2 \\ 1/3 & 0 & -1/2 \\ 1/3 & -1/2 & 1/2 \end{pmatrix}$ we find that it has three pivots, so the vectors are linearly independent. However, it is not an orthogonal basis: $\begin{pmatrix} 1/3 & 1/3 & 1/3 \end{pmatrix}\begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \end{pmatrix} \neq 0$.

4. Find the distance from $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ to the plane in $\mathbf{R}^3$ spanned by $v_1 = \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$.

Solution.

The distance is the length of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - proj_{span\{v_1,v_2\}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Note that $v_1 \cdot v_2 = 0$ so

$$proj_{span\{v_1,v_2\}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}}{\begin{pmatrix} 2 & -3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}} v_1 + \frac{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}}{\begin{pmatrix} 0 & 2 & -3 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}} v_2 = \frac{1}{17}v_1 - \frac{1}{13}v_2$$

Therefore, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - proj_{span\{v_1,v_2\}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-2/17 \\ 1+3/17+2/13 \\ 1-2/17+3/13 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. The distance is $\sqrt{a^2+b^2+c^2} \sim **$

5. If $y = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$, $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $u_2 = \begin{pmatrix} -1 \\ 6 \\ -5 \end{pmatrix}$. Find a decomposition $y = \widehat{y} + z$, where $\widehat{y}$ is in $W = Span\{u_1, u_2\}$ and z is in the orthogonal complement of W.

Solution.

We have that $\widehat{y} = proj_W(y)$ and that u_1 and u_2 are orthogonal. Thus

$$\widehat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1}u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2}u_2 = \frac{11}{3}u_1 + \frac{0}{u_2 \cdot u_2}u_2 = \frac{1}{3}\begin{pmatrix} 11 \\ 11 \\ 11 \end{pmatrix}.$$

Then $z = y - \widehat{y} = \frac{1}{3}\begin{pmatrix} -11 \\ 4 \\ 7 \end{pmatrix}$.