- **Combinatorics**
  is the study of arragements of objects
- **Enumeration**
  is the counting of objects with certain properties.

There are applications in

- Complexity of algorithms
- Probability
- Sequencing DNA

# Enumeration 101

Count ways to do a given task.

Math notation:

Want to find the cardinality of a set $|\mathcal{T}|$

where $\mathcal{T}$ is the set of all the ways to perform a given task.

# Examples

A class has 25 Software engineers and 30 Computer engineers. There are 18 $U2$-students and 10 U2-Software engineers.

Form a committee: President, Vicepresident and 3 secretaries.

- Ways to select a president: $25 + 30 = 55$
- Possible presidents in U2 or Software engineering: $18 + 25 - 10 = 33$
- Select president and vicepresident: $55 \cdot 54$
- Teams of secretaries: $\frac{55 \cdot 54 \cdot 53}{6}$

# Sum rule

If the task can be done either in one of $n_1$ ways or in one of $n_2$ ways; all of them distinct. Then in total there are

$n_1 + n_2$ ways to do the task.

Math notation: If $A \cap B = \emptyset$, then

$$|\mathcal{T}| = |A \cup B| = |A| + |B|$$

# Inclusion-Exclusion principle

For any sets $A_1, A_2, A_3$:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| = & |A_1| + |A_2| + |A_3| \\ & - [|A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1|] \\ & + [|A_1 \cap A_2 \cap A_3|] \end{aligned}$$

How would you get $|A_1 \cup A_2 \cup \cdots \cup A_n|$?

# Product rule

If the task can be broken down into a sequence of two tasks.
The first can be done in $n_1$ ways and the second in $n_2$ ways.
Then in total there are

$$n_1 \cdot n_2 \text{ ways to do the task.}$$

Math notation: Since $\mathcal{T} = A \times B$, then

$$|\mathcal{T}| = |A \times B| = |A| \cdot |B|$$

# Division rule

If the task can be done using a procedure that can be carried out in $n$ ways. And each outcome of such task is obtain from exactly $d$ distinct ways of that procedure. Then in total there are

$$\frac{n}{d} \text{ ways to do the task.}$$

Math notation: There is an equivalence relation in $\mathcal{P}$ (all classes of size $d$), then

$$|\mathcal{T}| = \frac{|\mathcal{P}|}{d}$$

# Permutations

Select $k$ distinct elements of a set of $n$ elements. Arrange them in a list so that the order of these elements matter.

Using the product rule (sequence of $k$ decisions), the total number of ways is

$$P(n, k) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

# Combinations

Select $k$ distinct elements of a set of $n$ elements. Arrange them in a set so that the order of these elements does not matter.

Using the division rule (forget the ordering in lists), the total number of ways is

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

# Examples revisited

A class has 25 Software engineers and 30 Computer engineers.
There are 18 $U2$-students and 10 U2-Software engineers.

Form a committee: President, Vicepresident and 3 secretaries.

- Ways to select a president: $25 + 30$
  Disjoint sets
- Possible presidents in U2 or Software eng: $18 + 25 - 10$
  Inclusion-Exclusion Principle
- Select president and vicepresident: $P(55, 2) = 55 \cdot 54$
  Sequence, order matters
- Teams of secretaries: $C(55, 3) = \frac{55 \cdot 54 \cdot 53}{3 \cdot 2 \cdot 1}$
  Equivalence, order does not matter

# Summary 1

- Sum rule (Disjoint sets) $|\mathcal{T}| = |A| + |B|$
- Inclusion-Exclusion Principle
  $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- Product rule (Sequence) $|\mathcal{T}| = |A| \cdot |B|$
- Division rule (Equivalence) $|\mathcal{T}| = |\mathcal{P}|/d$
- Permutations (Ordered) $P(n,k) = \frac{n!}{(n-k)!}$
- Combinations (Not ordered) $C(n,k) = \frac{n!}{k!(n-k)!}$

# Counting 201:
## Binomial coefficients and combinatorial proofs

# Combinations

The number $\frac{n!}{k!(n-k)!}$ has two interpretations:

$$\binom{n}{k} = \binom{n}{n-k}$$

Number of subsets of $k$ elements among a set of $n$ elements:

$$C(n,k) = C_k^n = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Or, num. of subsets of $n-k$ elements among a set of $n$ elements.

# Combinatorial Proofs

Use counting arguments to prove that both sides of an identity count the same collection of objects but in different ways.

Ex. Show that

$$\binom{n}{k} = \binom{n}{n-k}$$

# Example's proof

Consider the experiment of having $n$ **numbered balls**.
We will paint them so that there are $k$ blue balls and $n-k$ red balls. In **how many ways** can we do this?

- First pick the $k$ balls to be painted blue: $\binom{n}{k}$
- First pick the $n-k$ balls to be painted red: $\binom{n}{n-k}$

Both procedures give the same outcome, so

$$\binom{n}{k} = \binom{n}{n-k}$$

# Another example

Show that $36^6 - 26^6 = 10 \sum_{i=1}^{6} 26^{i-1} 36^{6-i}$

**Experiment:** The passwords on a computer system are *6 bits long*, are formed by *digits or uppercase letters* and *at least one digit*. How many distinct passwords are there?

- All strings (length 6) using digits and uppercases: $36^6$
  Substract (undesired) strings with no digits: $26^6$.
- Strings with first digit in $i$-th place: $10 \cdot 26^{i-1} \cdot 36^{6-i}$

# The Binomial theorem

Take $n$ factors:

$$(x+y)(x+y)\cdots(x+y).$$

In the expansion, how many terms $x^k y^{n-k}$ are there?
From which of $n$ parenthesis you pick $k$ terms equal to $x$?

Let $x, y \in \mathbb{R}$ and $n \geq 1$ be an integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}$$

Examples:

- $(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$;
- In $(x+y)^3$, the coefficient of $xy^2$ equals: $\binom{n}{3} = 3$

# Pascal's triangle: An important identity

Let $n \geq k \geq 1$ be integers, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

**Combinatorial proof.** Select $k$ of $n+1$ numbered balls.
Use sum rule considering two cases:

- Ball #1 is not selected; choose from $n$ balls: $\binom{n}{k}$.
- Ball #1 is selected; remains to select $k-1$ balls: $\binom{n}{k-1}$.

Or, just select $k$ of them: $\binom{n+1}{k}$.

# Pascal's triangle: An important identity

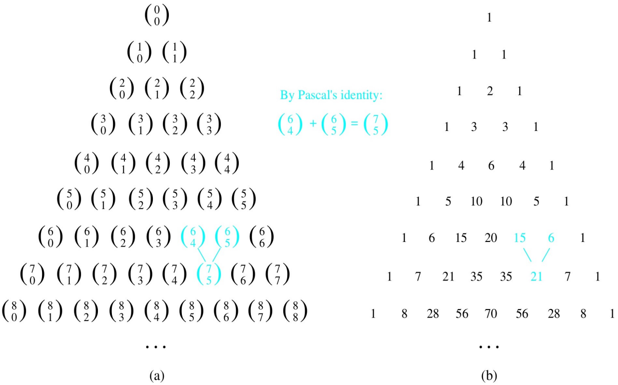


**FIGURE 1** **Pascal's Triangle.**

# Two Exercises

Let $n, k \geq 1$ be integers, then

$$\binom{n+k+1}{k} = \sum_{j=0}^{k} \binom{n+j}{j}.$$

**Hint:** Consider bit strings with a fixed number of zeros.

## Vandermonde's identity

Let $n, m \geq k \geq 1$ be integers, then

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}.$$

**Hint:** Pick a comittee of a given size, out of a group of men and women.

# The bars-and-stars trick

Select 5 bills from cash box with 7 slots for $1, $2, $5, $10, $20, $50 and $100 bills:

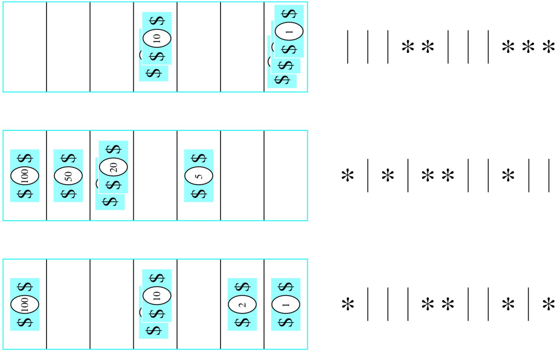


**FIGURE 2** **Examples of Ways to Select Five Bills.**

Imagine you combine 5 stars and 6 separating bars.

# The bars-and-stars trick

Select 5 bills from $\$1, \$2, \$5, \$10, \$20, \$50$ and $\$100$ bills:

$$\binom{5+6}{5} \text{ways.}$$

Same trick for:

- How many steps does the following sequence of loops has:

  $k := 0$
  for $i_1 := 1$ to $n$
  for $i_2 := 1$ to $i_1$
  $\vdots$
  for $i_m := 1$ to $i_{m-1}$
  $k := k + 1$

- Solutions to $x_1 + x_2 + x_3 = 30$; $x_1, x_2, x_3 \in \mathbb{N}$.

# Summary 2 (Table 1, Section 6.5)

| Select $k$ objects out of a set of $n$ elements | | | |
|---|---|---|---|
| Repetitions allowed | Ordered | Formula | Name |
| No | Yes | $\frac{n!}{(n-k)!}$ | Permutation |
| No | No | $\binom{n}{k}$ | Combination |
| Yes | Yes | $n^k$ | Permutation |
| Yes | No | $\binom{(n-1)+k}{k}$ | Combination |

**Important:** In exercises, **specify the case** you are using.

Discrete Probability $=$ Counting

# Laplace's definition of Probability

- **Experiment:** procedure that **yields an outcome**.
- Sample space: the set of possible outcomes.
- Event: a subset of the sample space.
  (usually defined by a given property of the outcome)

Let $S$ is a finite sample space of **equally likely outcomes**.
Then the probability of an event $E \subset S$ is

$$p(E) = \frac{|E|}{|S|} = \frac{\#\ \text{Favorable cases}}{\#\ \text{Total cases}}.$$

# Basic properties

Let $S$ is a finite sample space of **equally likely outcomes**.
Then the probability of an event $E \subset S$ is

$$p(E) = \frac{|E|}{|S|} = \frac{\#\text{ Favorable cases}}{\#\text{ Total cases}}.$$

Using the inclusion-exclusion principle:

- Complement: $p(\bar{E}) = 1 - p(E)$
- Union: $p(E \cup F) = p(E) + p(F) - p(E \cap F)$