Relations and functions

Let A and B be sets.

- A relation R from A to B is a subset $R \subseteq A \times B$.

  E.g. $A = \{2, 3, 5\}$ and $B = \{10, 11, \ldots, 15\}$

  $R_1 = \{(a, b) \in A \times B : \ b/a \in \mathbb{Z}\}$

  $R_2 = \{(2, 11), (2, 15), (3, 14)\}$

- A function F from A to B is a special type of relation.

  For every $a \in A$, F contains exactly one ordered pair (a, b).

  $F_1 = \{(a, b) \in A \times B : \ b = a + 9\}$

  $F_2 = \{(2, 11), (3, 14), (5, 14)\}$

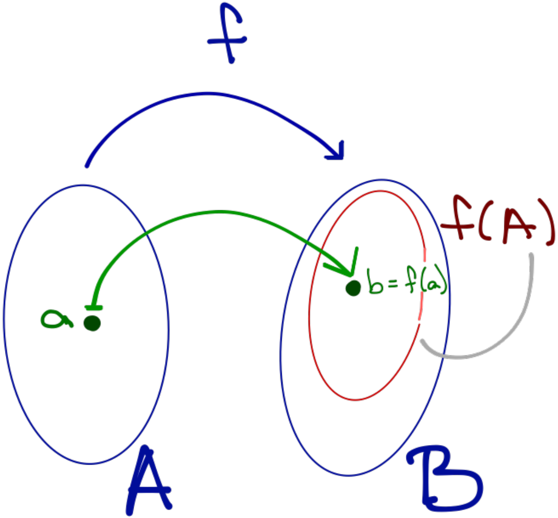
Standard notation for functions

A function F from A to B is a relation where, for every $a \in A$,

$$F \text{ contains exactly one ordered pair } (a, b).$$

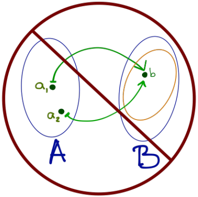
- $f : A \to B$, f maps A to B.
- Ordered Pairs: $(a, f(a)) = (a, b)$
  - b is image of a,
  - a is preimage of b.
- Domain: A
- Codomain: B
- Range: $f(A) = \{b \in B : b = f(a) \text{ for some } a \in A\}$

Standard notation for functions

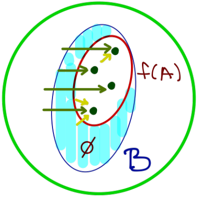


Types of functions

- **Injective**, *one-to-one*
  $f(a_1) = f(a_2)$ implies $a_1 = a_2$.



- **Surjective**, *onto*
  The range $f(A) = B$.



- **Bijection**, *one-to-one correspondence*
  Both injective and surjective.

Important functions

- Floor: $\lfloor x \rfloor$ is the largest integer which is $\leq x$,
- Ceiling: $\lceil x \rceil$ is the smallest integer which is $\geq x$,
- Exponential: for a positive integer n,

$$a^n = a \cdot a \cdot a \cdots a \text{ } (n \text{ times}),$$

- Factorial: for a positive integer n,

$$n! = 1 \cdot 2 \cdot 3 \cdots n,$$

Sequences

Sequences: are simply representations of functions

from $\mathbf{Z}^+$ to $\mathbf{R}$: $a_1, a_2, a_3, \ldots$
from $\mathbf{N}$ to $\mathbf{R}$: $a_0, a_1, a_2, a_3, \ldots$

- Arithmetic: $a_n = c + dn$
- Geometric: $a_n = c \cdot b^n$

Exercises with Summations and products

- $\sum_{i=1}^{10}(2+3i)$
- $\prod_{i=0}^{4}(3^i)$
- Convention: $\sum_{i=1}^{0} = 0$
- Convention: $\prod_{i=1}^{0} = 1$
- $\sum_{i=1}^{3}\prod_{j=i}^{i+4} j$

Concept of cardinality using functions

Two non-empty sets, A and B have the same cardinality if and only if there is a bijection from A to B.

Countable sets If there is a bijection between

- A and $\{1, \ldots, n\}$
  then $|A| = n$, S has cardinality n, finite.
- A and $\mathbf{N}$
  then $|A| = \aleph_0$, S has cardinality *aleph null*, infinite.

Uncountable sets

- If there is surjection from A to $\mathbf{N}$ but
  there is not a surjection from $\mathbf{N}$ to A.

Operations with functions

- Inverse: When $f : A \to B$ is a bijection.

$$f^{-1}(b) = a, \text{ where } f(a) = b$$

- Composition: When $g : A \to B$ and $f : B \to C$.

$$f \circ g(a) = f(g(a))$$

Extra: Operations with functions

- Sums: When $f_1, f_2 : A \to B$, and B is closed under sums.

$$(f_1 + f_2)(a) = f_1(a) + f_2(a)$$

- Product: When $f_1, f_2 : A \to B$, and B is closed under products

$$(f_1 f_2)(a) = f_1(a) f_2(a)$$

- Monotone(Increasing/Decreasing): When $f : A \to B$, and A, B are ordered sets.

$$a_1 \leq a_2 \text{ implies } f(a_1) \leq f(a_2)$$

Note: $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ are ordered sets and closed under sums and products.