Logical equivalence

Statements S and T are logically equivalent

$$S \equiv T$$

if and only if S and T have the same truth value no matter which

- predicates are subtituted into the statements,
- domain of discourse is considered.

Examples:

$$\exists x(P(x) \vee Q(x)) \quad \equiv \quad \exists x P(x) \vee \exists x Q(x)$$

$$\forall x(P(x) \vee Q(x)) \quad \not\equiv \quad \forall x P(x) \vee \forall x Q(x)$$

Think of $\exists$ as a ‘open-ended’ connector OR.

Think of $\forall$ as a ‘open-ended’ connector AND.

$$\begin{aligned}\exists x(P(x) \vee Q(x)) &\equiv \exists x P(x) \vee \exists x Q(x) \\ \forall x(P(x) \wedge Q(x)) &\equiv \forall x P(x) \wedge \forall x Q(x)\end{aligned}$$

$$\begin{aligned}\forall x(P(x) \vee Q(x)) &\not\equiv \forall x P(x) \vee \forall x Q(x) \\ \exists x(P(x) \wedge Q(x)) &\not\equiv \exists x P(x) \wedge \exists x Q(x)\end{aligned}$$

Negation of statements with quantifiers

S	$\neg S$
$\forall x\ \forall y P(x,y)$ $\forall y\ \forall x P(x,y)$	$\exists x\ \exists y\ \neg P(x,y)$ $\exists y\ \exists x\ \neg P(x,y)$
$\exists x\ \exists y\ P(x,y)$ $\exists y\ \exists x\ P(x,y)$	$\forall x\ \forall y\ \neg P(x,y)$ $\forall y\ \forall x\ \neg P(x,y)$
$\forall x\ \exists y\ P(x,y)$	$\exists x\ \forall y\ \neg P(x,y)$
$\exists x\ \forall y\ P(x,y)$	$\forall x\ \exists y\ \neg P(x,y)$

Exercises

- Every real number, has an additive inverse.

  For every real number x, there is a real number y such that $x + y = 0$.

  $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}, [x + y = 0]$.

- There exists an additive neutral elements for the real numbers.

  There exists a real number y such that for every real number x, $x + y = x$.

  $\exists y \in \mathbb{R}\ \forall x \in \mathbb{R}, [x + y = x]$.

Exercises

- For every two integer numbers, if these integers are both positive, then the sum of these integers is positive.

  $\forall x \in \mathbb{Z}\ \forall y \in \mathbb{Z}, [x + y > 0]$

- Every real number except zero has a multiplicative inverse.

  $\forall x \in \mathbb{R}^*\ \exists y \in \mathbb{R}\ , [xy = 1]$

  $\forall x \in \mathbb{R} \setminus \{0\}\ \exists y \in \mathbb{R}\ , [xy = 1]$

  $\equiv \forall x \in \mathbb{R}[x \neq 0 \to (\exists y \in \mathbb{R}, (xy = 1)]$

  $\equiv \forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ , [(x = 0) \vee (xy = 1)]$