Description of a set

A set is an unordered collection of objects which are called members or elements. To describe a set:

- List all its elements. $A = \{1, 2, 7, -3\}$
- State the properties objects have to satify to be members; we use elipses ($\ldots$) when the pattern for membership is clear.

$$
\begin{aligned}
N =& \{x : \ x \in \mathbb{Z}, \ x \geq 0\} \\
=& \{x \in \mathbb{Z} : \ x \geq 0\} \\
=& \{0, 1, 2, \ldots, \}
\end{aligned}
$$

We say x is contained in A ($x \in A$) if x is a member of A.

Subsets and equality of sets

- Subset: $A \subseteq B$ if and only if

$$\forall x(x \in A \rightarrow x \in B)$$

- Equality: $A = B$ if and only if

$$A \subseteq B \text{ and } B \subseteq A$$

- Proper subset: $A \subset B$ if

$$A \subset B \text{ and } A \neq B.$$

Special sets and names

- Universal set U, is the set containing all objects under consideration.
- Empty set $\emptyset$, is the set containing no elements.

  Note: For all sets A,

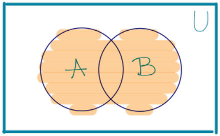
  $$A \subseteq U \qquad A \subseteq A \qquad \emptyset \subseteq A.$$

- Disjoint sets A and B, are sets with $A \cap B = \emptyset$.
- Singleton, is a set which contains exactly one element.

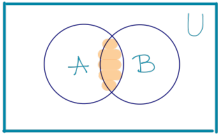
  Caution: $\emptyset \neq \{\emptyset\}$

Venn Diagrams and Operations

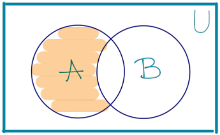
- Union
  $A \cup B = \{x : x \in A \lor x \in B\}$



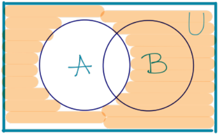
- Intersection
  $A \cap B = \{x : x \in A \land x \in B\}$



- Difference
  $A \setminus B = \{x : x \in A \land x \notin B\}$



- Complement
  $A^c = U \setminus A = \{x : x \notin A\}$



Notation: $\bar{A} = A^c$.

Generalized unions and intersections

- The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

$$A_1 \cup A_2 \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

- The intersection of a collection of sets is the set that contains those elements that are members of all sets in the collection.

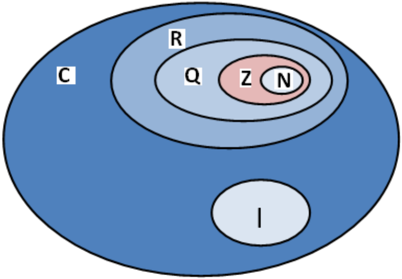
$$A_1 \cap A_2 \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

Examples

For each $i \in \mathbb{Z}$, let $A_i = \{i, i+1, i+2\}$ and $B_i = \{1, 2, \ldots, i\}$.

- $\bigcup_{i=1}^{5} A_i = \{1, 2, 3, \ldots, 7\}$,
- $\bigcap_{i=1}^{3} A_i = \{3\}$,
- $\bigcup_{i=1}^{n} B_i = \{1, 2, 3, \ldots, n\}$,
- $\bigcap_{i=1}^{n} B_i = \{1\}$.

Known sets



* *from http://www.onlinemath4all.com/*

$$N \subset Z^{+} \subset Z$$

Natural numbers:
$\mathbf{N} = \{0, 1, 2 \ldots\}$

Positive Integers:
$\mathbf{Z}^{+} = \{1, 2 \ldots\}$

Integers:
$\mathbf{Z} = \{\ldots, -1, 0, 1, 2 \ldots\}$

Rationals:
$\mathbf{Q} = \{\frac{p}{q} : p, q \in Z,\ q \neq 0\}$

Reals: $\mathbf{R}$

Complex: $\mathbf{C}$

Pure Imaginary numbers: $\mathbf{I}$

Sets can be elements in other sets

- Power set $\mathcal{P}(A)$; is the set containing all possible subsets of A

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

Example: If $A = \{a, b, c\}$, then

$$\begin{aligned}\mathcal{P}(A) = \{&\emptyset, \{a\}, \{b\}, \{c\}\\ &\{a, b\}, \{b, c\}, \{a, c\},\\ &\{a, b, c\}\}.\end{aligned}$$

Caution: $\{b, c\} \subseteq A$ and $\{b, c\} \in \mathcal{P}(A)$.

Sets vs. ordered n-tuples vs. strings

- A set S is an unordered collection of elements,

  if S contains n distinct elements then the cardinality of S is n. ($|S| = n$)
- An n-tuple $(a_1, \ldots a_n)$ is an ordered collection of elements. Where a_1 is the first element, ..., a_n is the n-th element.
- A string of length n $a_1 a_2 \cdots a_n$ is an ordered list of elements (or sequence of elements)

Note: 2-tuples are called ordered pairs.

Computer representation of sets

There are various ways to represent sets using a computer.

- We can store the elements of an set in an unorder fashion.

  But the operations $A \cup B, A \setminus B$ would be time-consuming.
  Searching for elements is required over and over.

- We can arbitrary fix an ordering of the elements in the universal set U.

  Then represent sets using strings of zeros and ones.

  Then set operations are equivalent to boolean algebra operations.

Exercises

Suppose the universal set is $U = \{1, 2, \ldots, 10\}$. Order the elements of U in increasing order, so that the i-th element is i. Then

- If $A = \{x : x \text{ is odd}\}$, then its bit string is 1010101010.
- If $B = \{x : x^2 \in U\}$, then its bit string is 1110000000.
- If $C = \{x^2 : x^2 \in U\}$, then its bit string is 1001000010.
- The bit string of A^c,

  0101010101
- The bit string of $B \setminus A$,

  0100000000
- The bit string of $C \cup A$,

  1011101010

Cartesian Product

Recall: an n-tuple $(a_1, \ldots b_n)$ is an ordered collection of elements.

$(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ if and only if $a_i = b_i \; \forall i \in \{1, \ldots, n\}$.

- The cartesian product of the sets $A_1, \ldots, A_n$ is

$$A_1 \times A_2 \times \cdots A_n = \{(a_1, \ldots, a_n) : \; a_i \in A_i \; \forall i \in \{1, \ldots, n\}\}.$$