MATH 363 Discrete Mathematics
Assignment 10

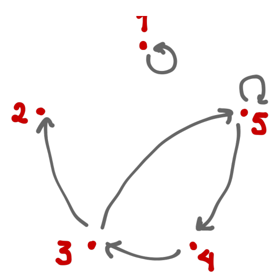
Due by March 30th

1. (**2pt** each) Let $A = \{1, \ldots, 5\}$. Give an example of a relation on A which
   - is transitive
   - is transitive but not reflexive
   - is reflexive but not symmetric
   - is antisymmetric

   List the elements, draw the digraph and matrix representing the relations. For example

$$R = \{(1,1), (3,2), (3,5), (4,3), (5,4), (5,5)\}$$

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$



2. (**2pt** each) Compute the compositions R^2 and R^3 of the relation R above, express them as a digraph and as a matrix.

3. (**2pt**) Prove that a relation is antisymmetric if and only if its digraph has no cycles of length 2.

4. (**2pt**) Fix a positive integer m. For any $a, b \in \mathbb{Z}$, we say that a is related to b if: $a = b$ **mod** m. Show that this defines an equivalence relation in $\mathbb{Z}$ and give the classes of these.

5. (**3pt**) Consider a graph $G = (V, E)$. For any $v, w \in V$, we say that v is related to w (write $v \sim w$) if $v = w$ or there is a path from v to w. Show that this defines an equivalence relation in V.

6. (**1pt** each) Draw a graph $G(V, E)$ with $|V| = 9$ and two vertices $v, w \in V$ in the same connected component; mark the following:
   i) A simple path from v to w of length 5,
   ii) A closed path going through v but which it is not a cycle,
   iii) A path from v to w which does not repeat edges but it does repeat vertices,

iv) The vertices adjacent to v;

v) What is the degree of w?

7. (**3pt** each) Consider a path $P = (x_0, x_1, x_2, \dots, x_n)$ connecting u to v

   i) Show that if $u \neq v$, then there is a simple path from u to v.

   ii) Show that if $u = v$ and there are no repeated edges in the path, then there is a subpath

   $$Q = (x_i, x_{i+1}, \dots, x_{k-1}, x_k)$$

   which is a cycle.

8. (**2pt**) Draw the graph G with adjacency matrix M, and give the partition of its vertices into connected components.

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

9. (**3pt**) Describe the strongly connected components and weakly connected components of R^3 in exercise 2. above.

10. (**2pt**) Use the handshaking theorem to show that any (undirected) graph has an even number of vertices of odd degree.

11. (**Extra: 3pt**) Give the pseudocode and apply Warshall's algorithm to the relation R above and give the digraph representing the resulting relation.