MATH 363 Discrete Mathematics
Assignment 3

Due by February 3rd

Some definitions to remember:
A non-negative integer n is a perfect square if there exists an integer k such that $n = k^2$.
The factorial numbers are defined as $n! = 1 \cdot 2 \cdot 3 \cdots n$ for all non-negative integers.

1. (**2pt**) Prove or disprove: if x is a nonzero real number, then $x^2 + 1/x^2 > 2$.
2. (**3pt**) Determine whether $\sqrt[3]{2}$ is rational.
3. (**4pt**) Prove that either $2 \cdot 3^{100} + 5$ or $2 \cdot 3^{100} + 6$ is not a perfect square.
4. (**3pt each**) Consider a student with major on Software Engineering and minor in French literature that is taking 6 classes this term.
   - All her midterms are going to be on the week (Monday-Friday) before Reading week. Show that she is going to have a day with two midterms.
   - Show that she is taking either 3 classes that are all related or 3 classes that have no relation between each other.
5. (**4pt**) How many distinct numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13\}$ to guarantee that at least one pair of these numbers add up to 14?
6. (**3pt**) Prove or disprove that you can use dominoes to tile the standard checkerboard with all four corners removed.
7. (**3pt**) Prove or disprove that you can use dominoes to tile the standard checkerboard with only two opposite corners removed.
8. (**1pt each**) Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for the positive integer n.
   - What is the statement $P(1)$?
   - Show that $P(1)$ is true, completing the basis step of the proof.
   - What is the inductive hypothesis?
   - What do you need to prove in the inductive hypothesis?
   - Explain why these steps show that $\forall n \in \mathbb{Z}^+ P(n)$ is true.
9. (**4pt**) Use the same steps as in the previous example to prove that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}.$$

10. (**4pt**) Prove using induction that $3^n < n!$ if n is a positive integer and $n > 6$.
11. (**2pt**) Explain what is wrong with the following proof:

- *Theorem*: For every positive integer n,

$$\sum_{i=1}^{n} i = \frac{(n+1/2)^2}{2}.$$

Proof. We will prove it by induction: The formula is true for $n = 1$, this is the basis step. Now, suppose that

$$\sum_{i=1}^{n} i = \frac{(n+1/2)^2}{2},$$

Then, using the inductive hypothesis on the second equality,

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= \left(\sum_{i=1}^{n} i\right) + (n+1) \\
&= \frac{(n+1/2)^2}{2} + (n+1) \\
&= \frac{(n^2+n+1/4)+2(n+1)}{2} \\
&= \frac{n^2+3n+9/4}{2} \\
&= \frac{(n+3/2)^2}{2} \\
&= \frac{[(n+1)+1/2]^2}{2}.
\end{aligned}$$

Thus, we have completed the inductive step and the theorem is true.