MATH 363 Discrete Mathematics
Assignment 4

Due by February 10th

1. (**1pt each**) Draw the Venn Diagram of the following sets.

   i) $[(A \cup B) \setminus C] \cup (A \cap C)$

   ii) $(A^c \cup B) \cap C$

2. (**1pt each**) Let $U = \{x \in Z : x^2 < 10\}$, represent the follows sets using bit strings. Use the increasing order to list the elements in U.

   i) $A = \{3, -2, 0, 1, -3\}$

   ii) $B = \{x \in U, \ x > 0\}$

   iii) $C = \{x \in U, \ |x - 1| < 2\}$

   iv) $(A \cap C)$

   v) $(A \cup B)$

   vi) $[(A \cup B) \setminus C] \cup (A \cap C)$

3. (**2pt each**) Let A, B be sets. Show that

   i) $(A \cap B) \subseteq (A \cup B)$

   ii) $A \cup B = A \cup (B \setminus A)$

4. (**2pt** each) Let $A = \{\emptyset, a, b, \{a, \{b\}\}\}$

   i) List all the elements of $B = \mathcal{P}(A)$

   ii) If $\{\emptyset\} \subseteq \mathcal{P}(C)$, what can you conclude about C? Justify your answer.

   iii) If $\{\emptyset\} \in \mathcal{P}(C)$, what can you conclude about C? Justify your answer.

5. (**3pt each**) Let A and B be distinct non-empty sets

   i) Show that $A \times B \neq B \times A$,

   ii) Give a bijection between $A \times B$ and $B \times A$.

6. (**3pt**) Show that if $A \cap C = B \cap C$ and $A \cup C = B \cup C$, then $A = B$

7. (**2pt** each) Let $A_i = \{1, 2, \ldots, i\}$. Give a description of the following sets.

   i) $A_{2i} \setminus A_{2i-1}$

   ii) $\displaystyle\bigcap_{i=1}^{40} (A_{2i} \setminus A_{2i-1})$

   iii) $\displaystyle\bigcap_{j=3i-1}^{3i+1} A_j$

   iv) $\displaystyle\bigcup_{i=2}^{4} \left(\bigcap_{j=3i-1}^{3i+1} A_j \right)$

8. (**2pt**) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \lceil x \rceil - \lfloor x \rfloor$. Show that $f(x) = 0$ if $x \in \mathbf{Z}$ and $f(x) = 1$ if $x \notin \mathbf{Z}$.

9. (**2pt**) Prove that for any non-empty countable set S, there exists a injective function from S to $\mathbf{N}$.

10. (**3pt**) Show that the set of points in the circle $C = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$ have the same cardinality that the points in the interval $I = [0, 1)$.

11. (**1pt**) Let P be the set containing the 5 pythagorean solids. Is P countable?

12. (**2pt each**) Let $f, g, h : P \to \mathbf{N}$ be functions such that, if $p \in P$ then

$$\begin{aligned} f(p) &= n && \text{if } p \text{ has } n \text{ faces} \\ g(p) &= m && \text{if } p \text{ has } m \text{ edges} \\ h(p) &= v && \text{if } p \text{ has } v \text{ vertices} \end{aligned}$$

For example, $f(\text{cube}) = 6$, $g(\text{cube}) = 12$, $h(\text{cube}) = 8$. (Justify your answers)

i) Are the functions f and g injective?

ii) Are the functions f and g surjective?

iii) What is the range of g and $f + h$?

13. (**2pt** each) Find the value of the following sums. Show your work.

i) $\displaystyle\sum_{i=1}^{40}(3 + 5i)$ *ii*) $\displaystyle\sum_{i=0}^{10}(1/3)^i$

14. (**3pt** each) Find the value of the following sums. Show your work.

i) $\displaystyle\sum_{i=2}^{4}\sum_{j=1}^{2i-3} 2j$ *ii*) $\displaystyle\sum_{i=2}^{4}\sum_{j=1}^{i}\lfloor 3j/2 \rfloor$

15. (**2pt**) Give an example of a function from $\mathbf{N} \to \mathbf{N}$ which is not injective and it grows at least exponentially (that is, is $\Omega(a^n)$ for some $a > 1$).

16. (**1pt each**) Determine whether the following statements are true.

i) $2 + x^3$ is $O(x^2)$

ii) $2 + x^3$ is $\Omega(x^2)$

iii) $\lfloor 2x \rfloor$ is $O(x)$

17. (**1pt each**) Give a big-O estimate for the following functions

i) $(n^3 + n^2 \log n)(3 - 2\log n)$

ii) $(n! - n^5)(3^n + 5^n)$

18. (**2pt each**) Use the definition of '$f(x)$ is $O(g(x))$, $\Omega(g(x))$, $\Theta(g(x))'$ to show that

i) $n^3 \log(n^2)$ is $\Omega(n^3)$,

ii) $\log(3^n)$ is $O(n)$,

iii) $n^3 + 5n^2 + 40$ is $\Theta(n^3)$.

19. (**2pt**) Describe an algorithm for finding both the largest and the smallest integers in a finite sequence of integers.

20. (**3pt**) Determine the worst-case performance of the algorithm above.

21. (**3pt**) Determine the least number of comparisons, or best-case performance required to find the maximum of a sequence of n integers (using the algorithm from class).