MATH 363 Discrete Mathematics
Assignment 9

Due by March 23rd

1. Consider the following experiment. First, flip a coin. If the coin lands head, throw 1 die. If the coin lands tails, then throw two dice. Let the sample space be $S = \{h, t\} \times \{1, 2, \ldots, 12\}$.

   i) (**1pt**) Give the set $E \subset S$ that represent the event that the sum of the dice is at least 5.

   ii) (**1pt**) Give the set $F \subset S$ that represent the event that the sum of the dice is even.

   iii) (**2pt**) If the coin and the dice are fair. What is the probability of F

   iv) (**2pt**) If the coin is twice as likely to land heads than land tails. What is the probability of E

2. *Monty Hall Three-Door contest* You have a chance to win a large prize. First, you are asked to select one of three doors to open; the large prize is behind one of the three doors and the other two doors are losers. Once you select a door, the game show host (who knows what is behind each door) opens one of the other two doors that he knows is a losing door (selecting at random if both are losing doors). Then he asks you whether you would like to switch doors.

   i) (**1pt**) What is the probability that you selected the winning door, in the first place?

   ii) (**2pt**) What is the probability that win the price if you switch doors?

3. A space probe near Neptune communicates with Earth using bit strings. Suppose that in its transmissions it sends a 1 one-third of the time and a 0 two-thirds of the time, independently of earlier transmittions. When a 0 is sent, the probability that it is received correctly is 0.9, and the probability that it is received incorrectly (as a 1) is 0.1. When a 1 is sent, the probability that it is received correctly is 0.8, and the probability that it is received incorrectly (as a 0) is 0.2.

   i) (**1pt**) What is the probability that the probe sent a 1 and Earth received a 1?

   ii) (**2pt**) What is the probability that Earth receives a 1.

   iii) (**2pt**) If Earth receives a 1, what is the probability that the probe sent 1?

   iv) (**2pt**) If Earth receives the string 11, what is the probability that the probe actually sent the string 11?

4. Consider a bijection $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$; such functions are called permutations on $[n]$. If $\sigma(i) = i$ we say that i is a fixed point of σ. We will obtain the probability that a random permutation on $[n]$ has no fixed points.

   i) (**1pt**) Let D be the event that σ has no fixed points, and F_i be the event that i is a fixed point in σ. Show that $\bar{D} = \cup_{i=1}^{n} F_i$.

   ii) (**2pt**) Use the generalized inclusion-exclusion formula to show that

   $$1 - P(D) = \sum_{i=1}^{n} P(F_i) - \sum_{1 \le i_1 < i_2 \le n} P(F_{i_1}, F_{i_k}) + \cdots \pm P(F_1, F_2, \ldots, F_n)$$

   iii) (**2pt**) Explain why, for $1 \le k \le n$ and $1 \le i_1 < \cdots < i_k \le n$, then

   $$P(F_{i_1}, \ldots F_{i_k}) = \frac{(n-k)!}{n!}.$$

iv) (**1pt**) Show that then

$$P(D) = \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

Note that the limit of $P(D)$ is e^{-1} as $n \to \infty$.

5. (**2pt**) If a permutation σ on $[n]$ is chosen at random. What is the expected number of fixed points?

6. I have 10 undistinguishable balls, I take each of them and, independently of the others, paint it red,black or pink with probability .3, .4 and .3 respectively.

   i) (**1pt**) If X counts the number of red balls; what is the distribution of X?

   ii) (**1pt**) If Y counts the number of black and red balls; what is the distribution of Y?

   iii) (**3pt**) What is the probability that I have at least one ball of each color?

7. I collect stamps from the cereal boxes. Suppose there are 5 different stamps and that each box has a uniformly chosen stamp out of the 5 possible ones.

   - (**1pt**) What is the probability that after opening 8 cereal boxes, I have at least 2 different stamps?
   - (**2pt**) If I already have 3 different stamps. What is the expected number of boxed I will have to open before I found a stamp I don't have already?

8. Generalize the problem above. Suppose I want to collect n stamps, and each box has a uniformly chosen stamp out of the n possible ones.

   i) (**2pt**) Given that I already have $i-1$ different stamps, let X_i be the number of boxes I have to open before I found a stamp I don't have already. What is the distribution of X_i

   ii) (**1pt**) What is the expected value of X_i?

   iii) (**2pt**) What does the sum of $X_1 + X_2 + \cdots X_n$ represent?