MATH 363 Discrete Mathematics
Assignment 10

SOLUTIONS

1 Grading Scheme

1. *i)-iv)* Out of **2pt** each: Full marks for an example.
   -.5pt for each representation which has a mistake.

2. Out of **2pt** each: Full marks if the correct relation is represented.
   -.5pt for each representation which has a mistake.

3. Out of **2pt**: 1pt if it is proven that an antisymmetric relation has no 2-cycles.
   1pt if it is proven that a relation with no 2-cycles is antisymmetric.

4. Out of **2pt**: 2/3pt if it is proven that the relation is symmetric.
   2/3pt if it is proven that the relation is transitive.
   2/3pt if it is proven that the relation is reflexive.

5. Out of **3pt**: 1pt if it is proven that the relation is symmetric.
   1pt if it is proven that the relation is transitive.
   1pt if it is proven that the relation is reflexive.

6. Out of **1pt** each: Full marks for correct example.

7. Out of **3pt** each: Full marks for a proof paraphrasing the ones below.
   2pt if there is an argument which is incomplete.
   1pt if there is only a particular example.

8. Out of **2pt**: 1pt for a correct drawing of the graph.
   1pt for a correct partition of the vertex set.

9. Out of **3pt**: Full marks if weakly connected components are correct.
   -.5pt for each strongly connected component which is not identified.

10. Out of **2pt**: Full marks if a paraphrase of the proof below is given.
    1pt if only an example is shown.

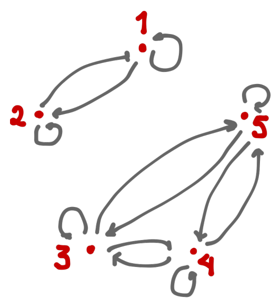
11. Out of **3pt**: 1pt for the correct pseudocode.
    .5pt for each step correct.

2 Assignment with solutions

1. (**2pt** each) Let $A = \{1, \ldots, 5\}$. Give an example of a relation on A which

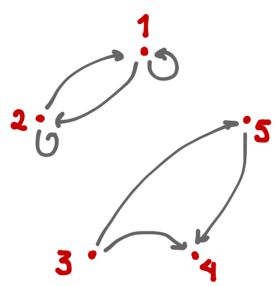
- is transitive
  $R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(3,5),(4,3),(4,4),(4,5),(5,3),(5,4),(5,5)\}$

$$M_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$



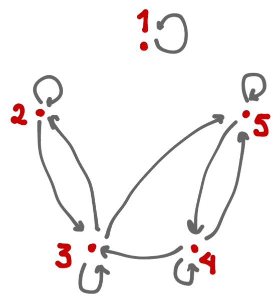
- is transitive but not reflexive
  $R_2 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(3,5),(5,4)\}$

$$M_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$



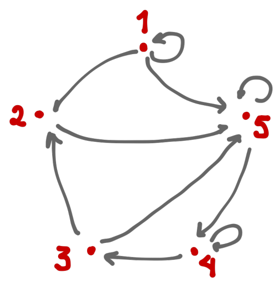
- is reflexive but not symmetric
  $R_3 = \{(1,1),(2,2),(2,3),(3,2),(3,3),(3,5),(4,3),(4,4),(4,5),(5,4),(5,5)\}$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$



- is antisymmetric R_4
  $R_4 = \{(1,1),(1,2),(1,5),(2,5),(3,2),(3,5),(4,3),(4,4),(5,4),(5,5)\}$

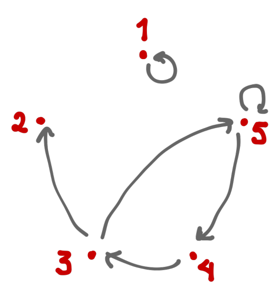
$$M_4 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$



List the elements, draw the digraph and matrix representing the relations. For example

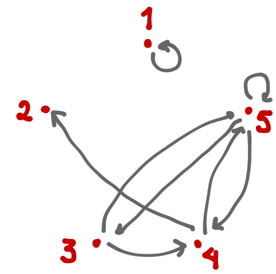
$$R = \{(1,1),(3,2),(3,5),(4,3),(5,4),(5,5)\}$$

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

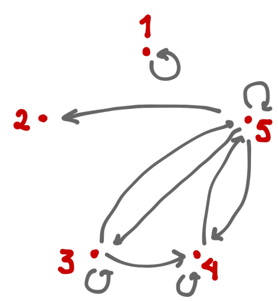


2. (**2pt** each) Compute the compositions R^2 and R^3 of the relation R above, express them as a digraph and as a matrix.

$$M_{R^2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$



$$M_{R^3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

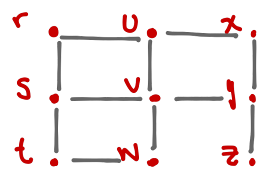


3. (**2pt**) Prove that a relation is antisymmetric if and only if its digraph has no cycles of length 2.
A cycle of length has the form $C = (x, y, x)$ where $x \neq y$. This means that $(x, y) \in R$ and $(y, x) \in R$ and $x \neq y$.
Thus, if a relation is antisymmetric, then there are no distinct elements x, y for which both (x, y) and $(y, x) \in R$; and therefore, there are no cycles of length 2.
It remains to prove that if the digraph is not antisymmetric, then there is a cycle of length 2. Assume then that R is not antisymmetric. Then there are two distinct elements x, y such that (x, y) and (y, x) are both in R, thus, the path $C = (x, y, x)$ is a cycle of length 2 in the digraph. The proof is complete.

4. (**2pt**) Fix a positive integer m. For any $a, b \in \mathbb{Z}$, we say that a is related to b if: $a = b$ **mod** m. Show that this defines an equivalence relation in $\mathbb{Z}$ and give the classes of these.
Reflexive: since $a - a = 0$ is divisible by any integer m. We have that $a = a$ **mod** m.
Symmetric: If $a = b$ **mod** m then $a - b$ is a multiple of m, and so does $b - a$; thus $b = a$ **mod** m
Transitive: If $a = b$ **mod** m and $b = c$ **mod** m then both $a - b$ and $b - c$ are multiples of m, and so does $(a - b) + (b - c) = (a - c)$; thus $a = c$ **mod** m.

5. (**3pt**) Consider a graph $G = (V, E)$. For any $v, w \in V$, we say that v is related to w (write $v \sim w$) if $v = w$ or there is a path from v to w. Show that this defines an equivalence relation in V.
Reflexive: By definition of $v \sim w$
Symmetric: If $v = w$ then symmetry is trivial. Suppose $v \neq w$ and $v \sim w$ then there is a path $P = (x_0 = v, x_1, x_2, \ldots, x_n = w)$. Now, the path $Q = (x_n = w, x_{n-1}, \ldots, x_1, x_0 = v)$ show that $w \sim v$
Transitive: If $v \sim w$, $w \sim u$ and all vertices are distinct then there are paths $P = (x_0 = v, x_1, \ldots, x_n = w)$ and $Q = (y_0 = w, y_1, \ldots, y_m = u)$. Therefore, there is a path from v to u: $S = (x_0, x_1, \ldots, x_n = y_0, y_1, \ldots, y_m = u)$. Similar arguments apply for the case when some of the three vertices are equal.

6. (**1pt** each) Draw a graph $G(V, E)$ with $|V| = 9$ and two vertices $v, w \in V$ in the same connected component; mark the following:



   i) A simple path from v to w of length 5,
   $P = (v, u, r, s, t, w)$

   ii) A closed path going through v but which it is not a cycle,
   $P = (s, v, y, x, u, v, w, t, s)$

   iii) A path from v to w which does not repeat edges but it does repeat vertices,
   $P = (v, u, r, s, v, w)$

   iv) The vertices adjacent to v;
   $N(v) = \{u, s, t, y\}$

   v) What is the degree of w?
   $\deg(w) = 2$

7. (**3pt** each) Consider a path $P = (x_0, x_1, x_2, \ldots, x_n)$ connecting u to v

   i) Show that if $u \neq v$, then there is a simple path from u to v.
   If P is a simple path then we are done. So suppose that there are two indices $0 \leq i < j \leq n$ with $x_i = x_j$. Then we can form the path $P_2 = (u = x_0, \ldots, x_i = x_j, x_{j+1}, \ldots, x_n = v)$ which still connects u to v (note that if $i = 0$ then $j \neq n$ and if $j = n$ then $i \neq 0$) and has a strictly shorter length. If P_2 is a simple path we are done. Otherwise, we repeat the process of finding shorter and shorter length. The process has to finish in at most n steps (you cannot reduce the length of a path with one edge), and we will have a simple path connecting u to v.

   ii) Show that if $u = v$ and there are no repeated edges in the path, then there is a subpath

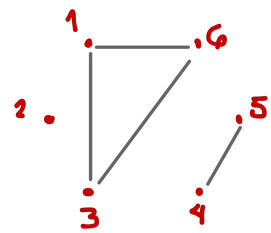
   $$Q = (x_i, x_{i+1}, \ldots, x_{k-1}, x_k)$$

   which is a cycle.
   If P is a cycle we are done. Otherwise there are two indices $0 \leq i < j \leq n$ with $x_i = x_j$. Now consider the path $Q_1 = (x_i, \ldots, x_j)$ this is a closed path; so either it is a cycle or we can repeat the process of finding two indices $i \leq k < l \leq j$ such that $x_k = x_l$ and consider the path $Q_2 = (x_k, \ldots, x_l)$. This process ends, also, because we cannot reduce the length of a closed path beyond 2.

Furthermore, since we are assuming that there are no edges repeated we will not have the case $Q = (x_i, x_{i+1}, x_{i+2} = x_i)$ and instead we will end with a closed path $Q = (x_i, \ldots, x_j)$ where $j \geq i + 3$ and no repeated vertices in the set $\{x_i, \ldots, x_j\}$.

8. (**2pt**) Draw the graph G with adjacency matrix M, and give the partition of its vertices into connected components.

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$



$V = \{1, 3, 6\} \cup \{2\} \cup \{4, 5\}$

9. (**3pt**) Describe the strongly connected components and weakly connected components of R^3 in exercise 2. above.
Weakly connected components: $\{1\}$ and $\{2, 3, 4, 5\}$
Strongly connected components: $\{1\}$ and $\{3, 4, 5\}$

10. (**2pt**) Use the handshaking theorem to show that any (undirected) graph has an even number of vertices of odd degree.
Suppose to the contrary that there is a graph with an odd number of vertices with odd degree. Let $V_1 = \{v \in V : \deg(v) = 1 \mathbf{mod} 2\}$ be the set of vertices with odd degree. Then,

$$\sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V \setminus V_1} \deg(v)$$

is a sum where the first term is odd and the second is even. This is a contradiction to the handshaking theorem, because the sum above is equal to $2|E|$, which is even. Thus, all graphs have an even number of vertices with odd degree.

11. (**Extra: 3pt**) Give the pseudocode and apply Warshall's algorithm to the relation R above and give the digraph representing the resulting relation.
The algorithm is given by:
procedure Warshall (M_R : $n \times n$ zeroone matrix)
$W := M_R$
for $k := 1$ **to** n
for $i := 1$ **to** n
for $j := 1$ **to** n
$w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return W

The matrix after the i-th iteration is W_i:

$$W_0 = M_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Vertex 1 has only a loop, so it can only connect 1 to 1 which is the same loop;

$$W_1 = M_R$$

There are no edges going out of 2;

$$W_2 = M_R$$

Vertex 3 can connect 4 to 2 and 4 to 5

$$W_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Vertex 4 can connect 5 to 2, 5 to 3 and 5 to 5 (the latter is a loop that already existed in the matrix);

$$W_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Vertex 5 can connect 3 to 3, 3 to 4, 4 to 4 and 5 to 5 (the latter is a loop that already existed in the matrix);

$$W_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The digraph corresponding to $W_5 = R^*$ is below