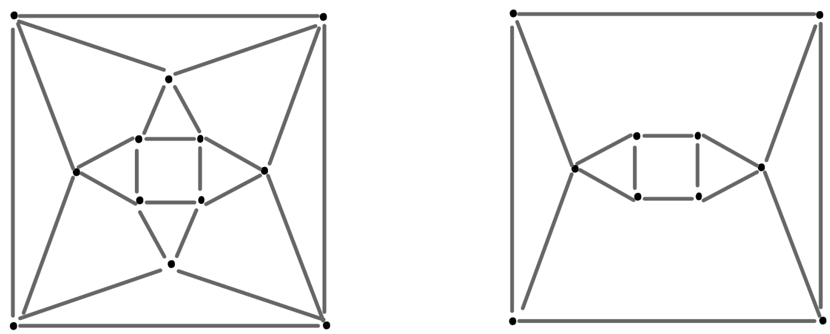
MATH 363 Discrete Mathematics
Assignment 11

SOLUTIONS

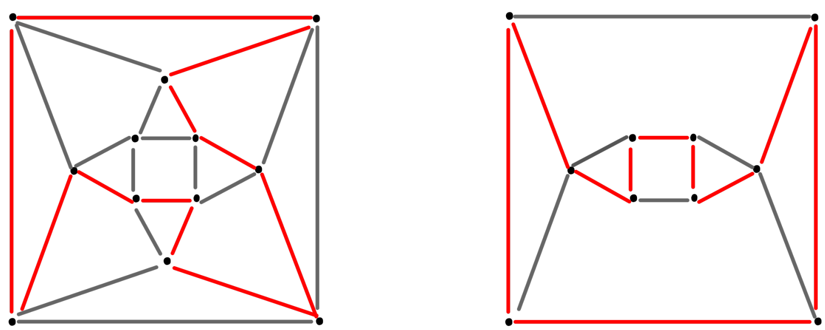
1 Grading Scheme

2 Assignment with solutions

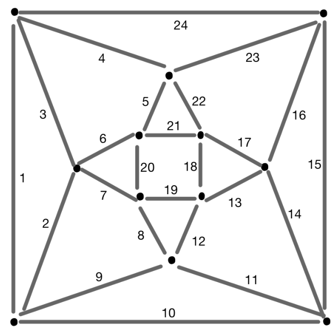
1. Consider the following two graphs:



i) (**2pt** each) Find a Hamiltonian path in each graph.
The following are Hamiltonian cycles, deleting one edge gives a Hamiltonian path



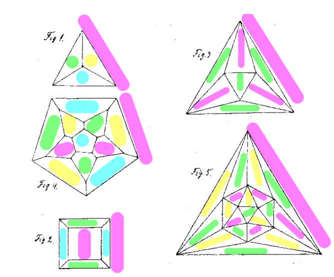
ii) (**2pt** each) Find an Eulerian path or cycle in each graph. If it is not possible, explain why.



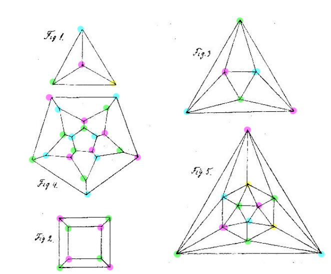
The second graph has no Eulerian cycle because it has vertices with odd degree; it does not have an Eulerian path either because it has more than 2 vertices with odd degree.

2. (**3pt**) Prove that if the minimum degree of a graph G is at least 2, then G has a cycle.
   Let $v_0 \in V$ be a vertex of the graph. Since it has degree at least 2, there exists at least one neighbour v_1. Now, we can find a vertex $v_2 \neq v_0$ which is neighbour of v_1, again, because we know that every vertex has at least 2 distinct neighbours. Similiarly, we can continue the sequence of vertices $v_3, \ldots, v_n$ such that both v_{i-1} and v_{i+1} are distinct neighbours of v_i. By the pigeonhole principle, there is a pair of repeated vertices in $\{v_0, \ldots, v_n\}$. Take the pair that has the smallest difference of indices, say $v_i = v_{i+j}$ (j is the difference of indices), then the path $P = (v_i, v_{i+1}, \ldots v_{i+j})$ has no repeated vertices and therefore it is a cycle.

3. (**2pt** each) Draw two of the 5 platonic solids as planar graphs, verify euler's formula for those graphs and color them with 4 colors.
   The following are face-colourings of the planar graphs that represent the platonic solids. Note that the external face needs to have a color too.



   The following are vertex-coloring of the graphs that represent the platonic solids. For these colourings it is not relevant that the graphs are planar.

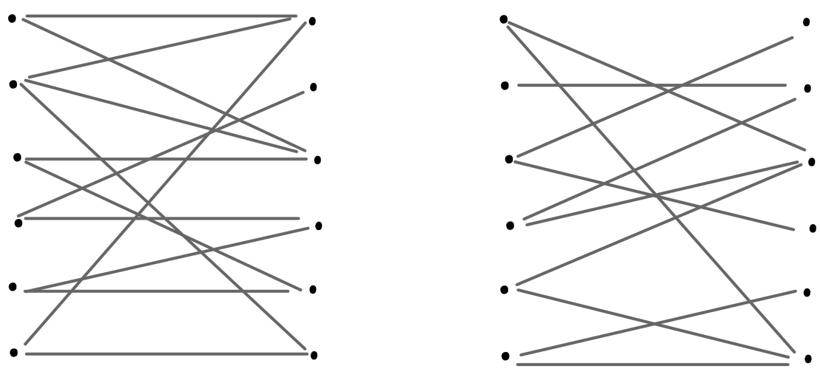


4. (**2pt**) Show that the n-cube is 2-colourable.
   Note: Showing a graph is 2-colourable is the same as showing that it is bipartite.

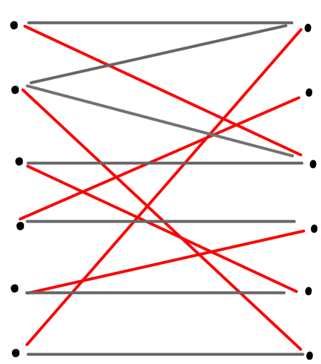
Recall that the vertices of an n-cube are $V = \{\text{bit strings of length } n\}$. We can partition $V = V_0 \cup V_1$ according to the number of ones in the bit string modulo 2; that is, $V_0 = \{x \in V : x \text{ has even number of ones}\}$ and $V_1 = \{x \in V : x \text{ has odd number of ones}\}$. For example if $n = 4$, $x = 1001$ and $y = 1101$ then $x \in V_0$ and $y \in V_1$.
Now we can colour vertices in V_0 with colour red, and vertices in V_1 with colour black. It will remain to show that all edges have one endpoint coloured red and the other black.
To see this, consider an edge $e = xy$ connecting vertices x and y. Then, by definition of an n-cube, we know that x and y differ only in one coordinate and so, the number of ones in x and y differ by exactly one. This means that each of them belong to distinct parts (V_0 and V_1) and so one is colour red and the other black. This holds for any edge, so the proof is complete.

5. (**2pt** each) Select two of the following applications to graph colorings; explain what is the problem and how it can be modeled and solved using graphs, give an example: Exam scheduling, Radio frequency assignments, index registers, solving sudoku puzzles.
See the textbook Section 10.8

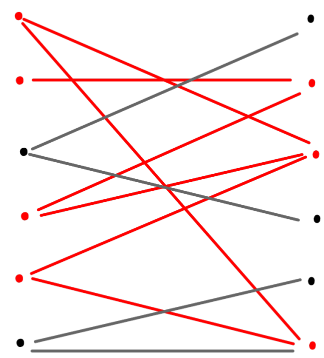
6. (**2pt** each) Given a graph $G = (V, E)$, a *complete matching* $M \subset E$ is a subset of the edges in G such that every vertex in V is incident to exactly one of the edges in M.
Find a complete matching for the following two graphs; if it is not possible, use Hall's theorem to prove it.



The first has a complete matching, for example



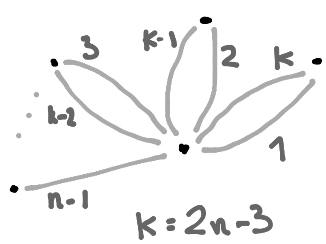
The second has no complete matching. If we number vertices on the left from top to bottom, let the set $A = \{1, 2, 4, 5\}$; A and its neighbourhood, $N(A)$, are marked with red below. By Hall's theorem, since there is a set A for which $|A| > |N(A)|$ there is no perfect matching in the graph.



7. Suppose there are n people in a group, each aware of a scandal no one else in the group knows about. These people communicate by telephone; when two people in the group talk, they share information about all scandals each knows about. For example, in the first call, each of these people knows about two scandals.

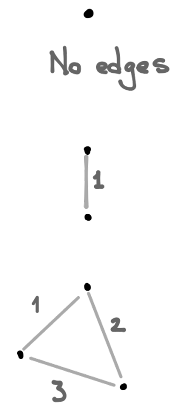
   i) (**1pt**) How many calls are used if we simply have every person call one person, a 'busy body', and then have that person call everyone back?
   $2n-3$ calls.
   Suppose people are represented by vertices $v_1, \dots, v_n$. In a first round, person v_1 calls everyone else ($n-1$ calls). Suppose that the last person to chat with v_1 is v_n, by that time v_1 knows all gossips from people $v_1, v_2, \dots, v_{n-1}$ and so v_n and v_1 know all gossips after their call.
   Now, in a second round, v_1 calls back $v_1, \dots, v_{n-1}$ to tell the rest of the gossips, this requires $(n-2)$ more calls.

   ii) (**1pt**) Represent the calls with a graph where edges are numbered in the order the calls are placed.



8. The gossip problem asks for $G(n)$, the minimum number of telephone calls that are needed for all n people to learn about all the scandals.

   i) (**1pt**) Compute $G(1), G(2)$ and $G(3)$; draw graphs with numbered edges to represent your solutions.
   $G(1) = 0, G(2) = 1$ and $G(3) = 3$.



ii) (**1pt**) Does the 'busy body' model above attains the minimum number of calls for $n = 4$. If not, give a better model.

No, $G(4) = 4$. Say v_1 and v_2 communicate. Then v_3 and v_4 communicate. Now, v_1 and v_3 communicate, and last: v_2 and v_4 communicate.

iii) (**2pt**) Prove by induction that $G(n) \leq 2n - 4$ for $n \geq 4$.

Hint: In the inductive step, have a new person call a particular person at the start and at the end.

The base step is given in the previous question: $G(4) \leq 4 = 2 \cdot 4 - 4$.

For the inductive step we use as hypothesis that $G(n) \leq 2n - 4$. That is, that a group of n persons can share all their gossips with at most $2n-4$ calls. Suppose now, that there are $n+1$, $v_1, v_2, \ldots, v_n, v_{n+1}$, each with one gossip to share.

One way to spread all rumors is the following: First, let v_{n+1} call v_1. Now, v_1 knows two gossips. Second, let $v_1, \ldots, v_n$ communicate with $G(n)$ calls and share all gossips. Now, each v_i, $1 \leq i \leq n$ knows all gossips, including the one from v_{n+1}. Last, let v_{n+1} call v_1 (or any other person) for it to learn the $n-2$ missing gossips. This means that

$$G(n+1) \leq G(n) + 2 = 2(n+1) - 4.$$

The proof by induction is complete.

iv) (**2pt**) Draw the graphs representing the inductive step above for $n = 5, 6$.