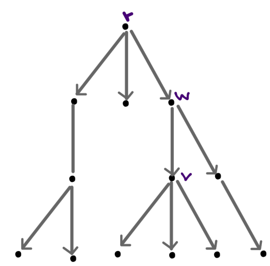
MATH 363 Discrete Mathematics
Assignment 12

SOLUTIONS

1 Grading Scheme

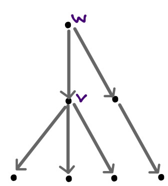
2 Assignment with solutions

1. (**1pt** each) Consider the following rooted tree T:



i) How many children has vertex v? 3

ii) Draw the subtree of vertex w.

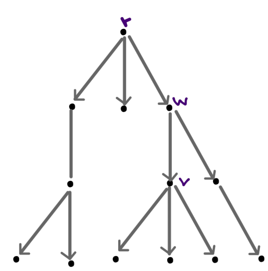


iii) What is the depth of vertex v? 2

iv) What is the height of of the tree? 3

v) Who is the parent of w? The root r

2. (**1pt**) Give an example of a 3-ary tree which is not a full 3-ary tree.



3. (**2pt**) What is the number of connected components in a forest with n vertices and m edges? $n - m$.
To see this, suppose that the forest $F = (V, E)$ consists of k trees $T_1 = (V_1, E_1), \ldots, T_k = (V_k, E_k)$. Each tree has $|V_i| = |E_i| + 1$ so

$$|V| = |V_1| + |V_2| + \cdots + |V_k| = (|E_1| + 1) \ldots + (|E_k| + 1) = |E| + k = m + k.$$

Thus, the number of trees is $k = n - m$.

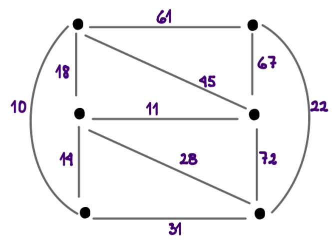
4. (**2pt**) Prove that trees are bipartite.
Solution 1: By induction on the number of vertices. Basis step: the tree with one vertex is trivially bipartite, there are no edges showing the contrary.
Inductive hypotesis: Trees with n vertices are bipartite.
Inductive step: Consider a tree $T = (V, E)$ with $n+1$ vertices, it has at least one leaf $v \in V$. Now, remove such vertex and its unique incident edge $e = wv$. The resulting tree $T' = (V \setminus \{v\}, E \setminus \{e\})$ is bipartite by the induction hypothesis. That is, there is a partition $A \cup B = V \setminus \{v\}$ of its vertices such that all edges in T' cross from set A to set B.
Suppose w.l.o.g. that $w \in A$. Then, consider the partition $V = A \cup B'$ with $B' = B \cup \{v\}$ of vertices in the original tree T. We claim that this show that T is bipartite. Every edge except $e = wv$ crosses from A to B' by the construction of $A, B($ $B \subset B')$; and also, the edge $e = wv$ crosses from A to B' by the definition of $B' = B \cup \{v\}$ and the assumption that $w \in A$. The proof is complete.
Solution 2: Let $T = (V, E)$ be a fix a vertex $v_0 \in V$ and consider T as a tree rooted at v_0. We know that the depth of a vertex is well defined, so color vertices with odd depth as red and vertices with even depth as black. Remains to show that there are no edges between vertices with the same color.
Suppose to the contrary that there are adjacent vertices v, w with the same color, say w.lo.g. black. Also, let's assume that v has the smallest depth (or equal): $2k$ (even because the color is black), now, because there is an edge connecting w to v, it has to be $2k+1$, but then w would had been coloured red. The contradiction comes from assuming that there was an edge connecting two vertices coloured black. Similarly, there are no two adjacent vertices coloured red.
Solution 3: A graph is bipartite if and only if it contains no cycles of odd length. Trees do not contain cycles, in particular no cycles of odd length. Therefore, trees are bipartite
Solution 4 (in fact, solution 3 extended):
Let $T = (V, E)$ be a fix a vertex $v_0 \in V$. Recall that, for any two distinct vertices v, w in a tree, there exists a unique simple path connecting v to w.
Now, consider the following subsets

$$V_0 = \{v_0\} \cup \{w \in V : \text{the simple path from } v_0 \text{ to } w \text{ has even length}\}$$
$$V_1 = \{v_0\} \cup \{w \in V : \text{the simple path from } v_0 \text{ to } w \text{ has odd length}\}$$

Since trees are connected, $V = V_0 \cup V_1$ is a partition of the vertices in the tree T.
It remains to prove that there are no edges with both endpoints in either V_0 (or V_1). Suppose to the contrary that there are two adjecent vertices v, w in V_0, let then there is a closed path $P = (v_0, \ldots, v, w, \ldots, v_0)$ composed of the simple path connecting v_0 to v, then going through the edge vw, and then using the simple path connecting w to v_0. This is a path of odd length. It can be proved that every closed path of odd length contains an odd cycle; this would be a contradiction since trees contains no cycles at all.

5. (**2pt**) A chain letter starts when a person sends a letter to five others. Each person who receives the letter either sends it to five other people who have never received it or does not send it to anyone. Suppose that 10,000 people send out the letter before the chain ends and that no one receives more than one letter. How many people receive the letter, and how many do not send it out?
This can be modelled with a full 5-ary tree, where internal vertices are people that received the letter and sent it to 5 others and external vertices are people that received the letter and did not sent it out. The number of letters sent are the number of edges in the tree
In this case, the number of internal vertices is $i = 10,000$. Thus, the number of edges in the tree is $m = 5 \cdot i$, the total number of vertices is $n = m + 1 = 5 \cdot i + 1$ and finally, the number of external vertices is $x = n - i = (5 - 1) \cdot i + 1$.
The number of people who received the letter is the total number of vertices except the root: 50,000.
The number of people who did not sent the letter out is the number of external vertices: 40,001.

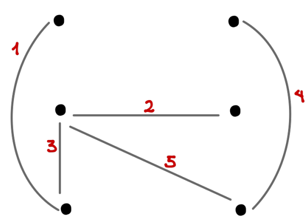
6. Consider the following graph G with edge weights as indicated below.



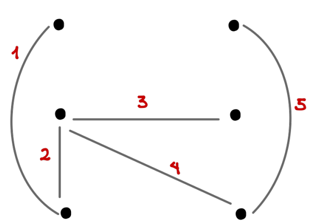
i) (**3pt**) Apply Kruskal's algorithm to G. Show all the steps of the algorithm.

ii) (**3pt**) Apply Prim's algorithm to G. Show all the steps of the algorithm.

KRUSKAL's



PRIM's



7. (**3pt** each) Choose two of the following type of trees, and write a short introduction about them: definition, motivation and examples: Binary Search Trees, Decision trees, Game trees.
See the textbook Section 11.2