MATH 363 Discrete Mathematics
Assignment 2

Due by January 27th

1 Grading Scheme

1. Full marks if the compound proposition is equivalent to the answer below.

   -3pt if propositional variables are not clearly defined.

2. +1pt if stated NOT CONSISTENT

   - +2pt If argument uses a truth table.
     -.5pt for each assignment of truth values which has no unsatisfied specification marked (specifications that are satisfied need not to be 'declared')
   - +2pt If argument uses proof by contradiction similar to solution 2.
     -1pt if there are invalid arguments used.

3. Full marks if answered as below, and

   +1pt if answer is logically equivalent.

4. Full marks if answered, then

   -1pt if intial propositional variables are misplaced,

   -1pt if there is one a logical connector misdrawn,

   -2pt if there is more than one logical connector misdrawn.

5. Full marks if answered, then

   -1pt if quantifiers are miswritten,

   -1pt if negation of the propositional function is incorrect,

   -1pt if counterexample is not valid. For *iii*), it must be stated that there is no counterexample, since the (negation of the) statement is true.

6. Full marks if answered, then

   -1pt if domain any propositional function is not clear,

   -1pt if parenthesis in expression are missing or misplaced,

   -1pt if propositions and/or propositional variables are not clearly defined,

   -1pt if expression does not convey the idea of the statement.

7. +.5pt pt if it stated NOT LOGICALLY EQUIVALENT

   - If it shows a counterexample:
     +.5pt if the counterexample is right
     +1pt if it explains clearly why it is, in fact, a counterexample
   - +1.5pt if it gives solution 4 clearly.

2 Assignment solutions

1. (**2pt each**) Express each of these specifications using propositional logic (Hint: define 3 propositions and write the statements below in terms of the propositions you define.)

   m**:** The system is in multiuser state.

   k**:** The kernel is functioning.

   t**:** The system is in interrumpt mode.

   i) If the system is in multiuser state, the kernel is functioning.
   $m \to k$

   ii) The kernel is not functioning or the system is in interrupt mode.
   $\neg k \vee t$

   iii) If the system is not in multiuser state, then it is in interrupt mode.
   $\neg m \to t$

   iv) The system is not in interrupt mode.
   $\neg t$

   $\rightsquigarrow$ **Definition:** A collection of (compound) propositions is *consistent* if there is an assignment of truth values to each of the proposition variables involved such that every proposition in the collection is true.

2. (**3pt**) Determine whether the specifications in the previous exercise is a consistent collection of propositions.

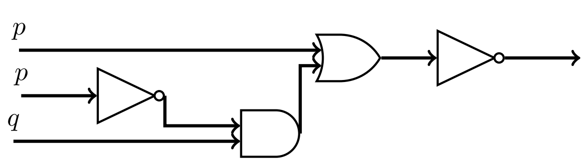
   Solution 1: There is no assignment of truth values for m, k, t which makes all specifications true. That is, in every row in the truth table we can find one proposition which is false.

m	k	t	$m \to k$	$\neg k \vee t$	$\neg m \to t$	$\neg t$
T	T	T				F
T	T	F		F		
T	F	T	F			F
T	F	F	F			
F	T	T				F
F	T	F		F	F	
F	F	T				F
F	F	F			F	

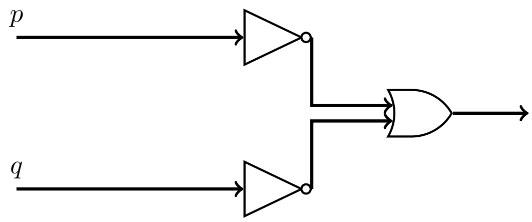
   Solution 2: First, the collection of specifications are logically equivalent to $m \to k$, $k \to t$, $\neg t \to m$ and $\neg t$. If the specifications are consistent it means that the propositions above are all true. Then, $\neg t$ being true and $\neg t \to m$ being true implies that m is true. This in turn, together with $m \to k$ being true implies that k is true, since $k \to t$ is also assumed to be true implies that t is true. This gives a contradiction since t and $\neg t$ cannot be simultaneously true. Therefore, the system of specifications can not be true.

3. (**3pt each**)Find out the output of these logic circuits.

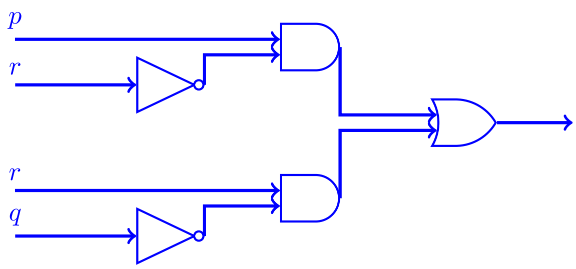
   i) $\neg[p \vee (\neg p \wedge q)]$ $(\equiv \neg p \wedge \neg q)$



   ii) $\neg p \vee \neg q$ $(\equiv p \to \neg q \equiv q \to \neg p)$



4. (**3pt each**) Construct a logic circuit using inverters, OR gates, and AND gates that produces the output $(p \wedge \neg r) \vee (\neg q \wedge r)$ from input bits p, q, and r.



5. (**3pt each**) Negate the statements, and find a counterexample to either the statement or its negation.
Note: Only false statements including the quantifier $\forall$ have counterexamples

   i) $\forall x \in \mathbb{R}\,(|x| > 0)$ This is false, a counterexample is $x = 0$ since $|x| \leq 0$
   $\exists x \in \mathbb{R}\,(|x| \leq 0)$

   ii) $\forall$ integer $x > 4$, $(x^2 \leq 10)$ This is false, a counterexample is $x = 5$ since $x^2 = 35 > 10$
   $\exists$ integer $x > 4$, $(x^2 > 10)$

   iii) $\exists x \mathbb{R}\ \forall y \in \mathbb{R}\,(xy = 1)$
   $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\,(xy \neq 1)$. This is true: for any $x \in \mathbb{R}$ let $y = 0$; then $xy = 0 \neq 1$.

6. (**4pt each**) Express each of these system specifications using predicates, quantifiers and logical connectives.

   i) No directories in the file system can be opened and no files can be closed when system errors have been detected.

   Solution 1: $(\exists z\, P(z)) \rightarrow \forall x\, \forall y\, (\neg O(x) \wedge \neg C(y))$
   Domain for $O(x)$: Directories in the file system
   Domain for $C(y)$: Files in the file system
   Domain for $E(z)$: System errors

   $O(x) : x$ can be open.

   $C(y) : y$ can be closed.

   $P(z) : z$ has been detected.

   Solution 2: $(\exists z\, (P(z))) \rightarrow \forall x\, \forall y\, (\neg O(x) \wedge \neg C(y))$
   Domain for $O(x)$: Directories, files and system errors

   $O(x) : x$ is a directory in the file system and it can be open.

   $C(y) : y$ is a file in the file system and it can be closed.

   $P(z) : z$ is a system error and it has been detected.

   Solution 3: $(\exists$ Error system $z\, P(z)) \rightarrow \forall$ Directories $x\, \forall$ Files $y\, (\neg O(x) \wedge \neg C(y))$

   $O(x) : x$ can be open.

   $C(y) : y$ can be closed.

$P(z)$: z has been detected.

ii) The file system cannot be backed up if there is a user currently logged on.

Solution 1: $(\exists y\, L(y)) \to \forall x\, (F(x) \to \neg B(x))$
Domain for $F(x), B(x)$: Systems
Domain for $L(y)$: Users

$F(x)$: x is a file system

$B(x)$: x can be backed up

$L(y)$: y is currently logged on.

Solution 2: $(\exists y\, L(y)) \to B$
Domain for $L(y)$: Users

B : The file system can be backed up

$L(y)$: y is currently logged on.

Note: It assumes that there is only one file system.

7. (**2pt**) Determine whether $\forall x(P(x) \leftrightarrow Q(x))$ and $\forall x P(x) \leftrightarrow \forall Q(x)$ are logically equivalent. Justify your answer.

Solution 1: Let

$$S \equiv \forall x(P(x) \leftrightarrow Q(x))$$
$$T \equiv \forall x P(x) \leftrightarrow \forall Q(x);$$

we will present a counterexample to show that S and T are not logically equivalent.
Consider the integers as the domain of discourse,

$$P(n) : n \text{ is even and } Q(n) : n+1 \text{ is even.}$$

With these choices we will show that S is false and T is true.
First, it is clear that the truth values of $P(n)$ and $Q(n)$ are opposite no matter the value of n, which implies that $P(n) \leftrightarrow Q(n)$ is false for all integers n. Thus, for this particular choice of domain and propositions $P(n), Q(n)$ we have that statement S is false.
On the other hand, it is also clear that each of the statements $T_1 \equiv \forall n\, P(n)$ and $T_2 \equiv \forall n\, Q(n)$ are both false. This implies that, for this particular choice of domain and propositions $P(n), Q(n)$, the statement T is true.

Solution 1b: Another counterexample:

Let the domain be the reals, $P(x) : x^2 > 0$ and $Q(x) : x = 0$. As in the first solution the truth values of $P(x)$ and $Q(x)$ are opposite for all real numbers x. The argument follows as above.

Solution 3: Let

$$S \equiv \forall x(P(x) \leftrightarrow Q(x))$$
$$T \equiv \forall x P(x) \leftrightarrow \forall Q(x);$$

we will present a counterexample to show that S and T are not logically equivalent.

Let the domain be the reals, $P(x) : x^2 > 100$ and $Q(x) : x < 0$.

It is clear that each of the statements $T_1 \equiv \forall x\, P(x)$ and $T_2 \equiv \forall x\, Q(x)$ are both false. This implies that, for this particular choice of domain and propositions $P(x), Q(x)$, the statement T is true.

Now, we will show that the statement S is false by showing a counterexample to S: it suffices to find a real number c, say $c = -15$ such that $P(c)$ is false and $Q(c)$ is true. (for the choice of domain and propositional functions $P(x), Q(x)$, any real number smaller that -10 is a counterexample to S). This show that S is false.

Solution 4: To be written.